 Printed in Belgium

Optimal Economic Growth and Stationary Ordinal Utility —A Fisherian Approach

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Received October 28, 1970

I. INTRODUCTION AND SUMMARY

The theory of optimal economic growth, which was started by Frank Ramsey more than forty years ago is now establishing its position as an important branch of modern welfare economics.[1] It is the theory that investigates the problem of capital accumulation—the problem of intertemporal resource allocation—from the normative point of view.

This welfare theory is set up as a constrained maximization problem: The objective function is an intertemporal utility function which summarizes the society's time preference structure, and the constraints are capital accumulation equations which describe the supply conditions of consumption goods and capital goods. Therefore, the nature of an optimal program is crucially dependent not only on the technological conditions but also on the properties of the intertemporal utility function. However, recent developments of the optimal growth theory have neglected the importance of the subjective utility function and almost taken for granted Ramsey's additive utility function. The purpose of this paper is to release the conventional optimal growth theory from this very restrictive utility function.[2] Instead we shall present an optimal growth theory that is based upon the more general *stationary ordinal utility function* introduced by Tjalling Koopmans.[3] This type of ordinal utility function includes the Ramsey type utility as a special case.

We shall first derive necessary conditions for optimal capital accumu-

[1] See Ramsey [12], Cass [3] and Koopmans [7].

[2] An independent study by Beals–Koopmans [1] deals with a similar model from a different viewpoint. (The author is indebted to the referee for this reference.) Another attempt is made by Uzawa in his [16] and [17] to introduce flexible discount rates into additive utility functions. See also Samuelson [15].

[3] See Koopmans [5] and Koopmans–Diamond–Williamson [6].

lation which require that a marginal rate of time preference must be brought into equality with a marginal rate of return over cost in each period. They might be regarded as a normative interpretation of Irving Fisher's equilibrium condition for intertemporal resource allocation.[4] However, since the marginal rate of time preference is generally dependent on a perpetual stream of future consumption, this Fisherian condition becomes an infinite-order difference equation.

We shall then solve this formidable problem by applying *the principle of dynamic consistency* of an optimal program, a generalization of the well-known principle of optimality in the dynamic programming. By virtue of this principle, we can reduce the search for an optimal program to a succession of single-period choices between an immediate consumption and an investment in capital stock. This trick is made possible by introducing two novel concepts—*the maximum welfare function* and *the reduced utility function*. The former function specifies a one-to-one relationship between an initial capital stock and the economy's maximum attainable value of intertemporal utility, and the latter summarizes the economy's single-period preference between an immediate consumption and an investment in capital stock. When the stationary ordinary utility function is strictly quasi-concave and the production function is concave, we are able to prove strict quasi-concavity of the reduced utility function. This property and continuity of the reduced utility function will turn out to be essential in our subsequent discussion. (In the Appendix, we shall discuss a successive approximation method to determine the functional forms of both the maximum welfare function and the reduced utility function solely from the given production function and intertemporal utility function. We shall also show their continuity in the Appendix.)

The resulting optimality condition implies that an optimal program must equalize a marginal rate of technological transformation between capital and consumption with a marginal rate of substitution between them defined in terms of the reduced utility function. Continuity and strict quasi-concavity of the reduced utility function assure us that the accumulation program that satisfies the above tangential condition in each period is indeed the unique optimal program that our planning authority is looking for.

The principle of dynamic consistency will enable us to illustrate the generation of optimal programs by simple two-dimensional Fisherian diagrams.[5] It will be shown that the optimal program for any initial capital stock must move along an Engel curve (or income-consumption line) of the reduced utility function in our Fisherian diagram. Moreover, asymptotic behaviors of optimal programs, in particular, their monotonicity and stability will be seen to be deeply related to normality/neutrality/inferiority of capital and consumption defined in terms of our reduced utility function.

In our optimal growth model the so-called turnpike property of optimal programs still holds but in a somewhat undermined way. Although the position of zero-capital and zero-consumption can be counted as a generalized turnpike, we can prove neither existence nor uniqueness of a nonzero generalized turnpike on which a positive level of consumption is sustainable.

If both capital and consumption are assumed to be normal goods for the economy, both optimal capital sequences and optimal consumption sequences are monotone in time. But even in this well-behaved case, there may not exist a nontrivial turnpike, and even if it exists there can be multiple nontrivial generalized turnpikes alternately stable and unstable. If consumption becomes inferior in some period, optimal consumption sequences cease to be monotone and a generalized turnpike with inferior consumption becomes necessarily an unstable one. Furthermore, if capital stock becomes inferior in some period, optimal programs begin cobweb type cyclical movements around a generalized turnpike from then on. When inferiority of capital is relatively weak, this oscillatory optimal program will be dampened and converge to the generalized turnpike. But when inferiority is relatively strong, the optimal program will either diverge spirally from the turnpike or approach a perpetual periodic motion. In any case, optimal programs will approach one of generalized turnpikes including the trivial one or converge to a periodic motion in the long-run, but their asymptotic behaviors will be much more complicated than in the standard Ramsey model.

Our optimal growth theory may be regarded as a generalization of the Ramsey theory of optimal saving, as well as a dynamization of Irving Fisher's geometrical theory of capital.

[4] See Fisher [4].

[5] See Part III in Fisher [4] and Samuelson [14]. This paper may shed new light on the controversy between Leontief [9, 10] and Westfield [18], which argues the possibility of an extension of the Fisherian graphical exposition of the two-period paths to programs involving many periods. (After the first version of the present paper was completed, the author came across a paper by Liviatan [11] which develops a Fisherian diagrammatic technique similar to ours to analyse the optimal growth path in the case of Ramsey type utility functions.)

II. The Basic Model

Our point of departure is a neoclassical one-sector model of production:

$$\begin{aligned} &K_t - K_{t-1} + C_t \leqslant F(K_{t-1}, N_{t-1}) - \delta K_{t-1}, \\ &C_t \geqslant 0, \qquad K_t \geqslant 0, \qquad (t = 1, 2, 3, \ldots), \end{aligned} \tag{1}$$

where K_t is capital stock available at the end of period t, N_t is labor force in period t and C_t is an amount of consumption goods produced in period t, respectively. It is assumed that the rate of capital depreciaiton δ is positive and constant for all t. The production function $F(K, N)$ is linearly homogeneous and subject to the usual neoclassical conditions.

Assuming that labor force is exponentially growing at a constant rate $n \geqslant 0$,

$$N_t = (1 + n)^t, \tag{2}$$

we can rewrite (1) in per capita form:

$$k_t + c_t \leqslant g(k_{t-1}), \tag{3}$$

where $c_t = C_t/N_t$, $k_t = K_t/N_t$ and

$$g(k) = \{F(k, 1) + (1 - \delta)k\}/(1 + n). \tag{4}$$

The technological function $g(k)$ defined above is assumed to be concave and possesses the following neo-classical properties (see Fig. 1):

$$g(0) = 0, \qquad 0 < g(k) - k < +\infty \qquad \text{for } \ 0 < k < \bar{k} < +\infty, \tag{5}$$

where $g(\bar{k}) - \bar{k} = 0$.

$$0 < g'(k) = \{F_K(k, 1) - \delta + 1\}/(1 + n) \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 1 \Leftrightarrow \begin{cases} 0 \leqslant k < k^\dagger \\ k = k^\dagger \\ k^\dagger < k \leqslant \bar{k} \end{cases} \tag{6}$$

$$\lim_{k \to 0} g'(k) = +\infty. \tag{7}$$

$$g''(k) = F_{KK}(k, 1) < 0 \qquad \text{for any } 0 \leqslant k \leqslant \bar{k}. \tag{8}$$

$\bar{k}$ is the maximum feasible capital stock that is the upper boundary of all the feasible capital stocks, and $k^\dagger$ is the Golden rule point or the production Bliss point *à la* Ramsey on which the marginal net productivity of capital $g'(k) - 1$ is zero. As is shown in Fig. 1, the curve $c = g(k) - k$ is a hill-shaped curve whose summit is the Golden rule point $k = k^\dagger$.

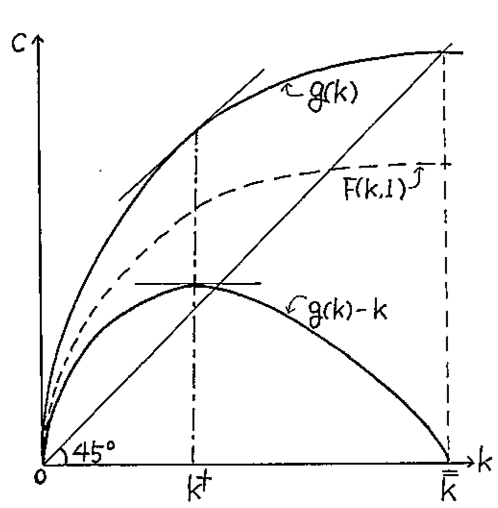

FIG. 1. Technological schedules in the economy.

Let us define an intertemporal utility function with infinite time horizon in order to evaluate a perpetual per capita consumption sequence ${}_1C$.[6]

$$U_1 = U(c_1, c_2, c_3, \ldots) = U({}_1C). \tag{9}$$

Instead of specifying a functional form of (9) *a priori*, we assume the following four postulates concerning the nature of the society's intertemporal preference ordering;

(P-1) Existence and continuity of a utility function over the feasible set of consumption sequences.[7]

(P-2) Sensitivity: there exist initial period consumptions c_1, c_1' and a sequence ${}_2C$ from the next period on, such that

$$U(c_1, {}_2C) > U(c_1', {}_2C). \tag{10}$$

(P-3) Limited noncomplementarity among periods: For all c_1, c_1', ${}_2C$ and ${}_2C'$,

$$U(c_1, {}_2C) \geqslant U(c_1', {}_2C) \ \text{ implies } \ U(c_1, {}_2C') \geqslant U(c_1', {}_2C'), \tag{11a}$$

$$U(c_1, {}_2C) \geqslant U(c_1, {}_2C') \ \text{ implies } \ U(c_1', {}_2C) \geqslant U(c_1', {}_2C'). \tag{11b}$$

[6] We shall denote a per capita consumption sequence $(c_1, c_2, c_3, \ldots)$ by ${}_1C$ and its truncated sequence $(c_t, c_{t+1}, c_{t+1}, c_{t+2}, \ldots)$ by ${}_tC$. We shall also denote by ${}_0K$ a per capita capital sequence $(k_0, k_1, k_2, \ldots)$ and by ${}_tK$ its truncated sequence $(k_t, k_{t+1}, k_{t+2}, \ldots)$.

[7] See Koopmans–Diamond–Williamson [6] for the precise definition of this postulate. In this paper, however, continuity is defined in terms of the product topology.

(P-4) Stationary: for some c_1 and all ${}_2C$ and ${}_2C'$,

$$U(c_1, {}_2C) \geqslant U(c_1, {}_2C') \quad \text{if and only if} \quad U({}_2C) \geqslant U({}_2C'). \tag{12}$$

Then, according to Koopmans [5], there exists a scalar function $V(c, U)$ such that the utility function (9) satisfies the following relation:

$$\begin{aligned} U_1 = U({}_1C) &= V(c_1, U({}_2C)) = V(c_1, V(c_2, U({}_3C))) \\ &= V(c_1, V(c_2, V(c_3, V(..., U({}_TC))))), \quad (T = 4, 5, 6,...). \end{aligned} \tag{13}$$

This relation can be replaced by a recurrent formulation

$$U_t = V(c_t, U_{t+1}), \qquad (t = 1, 2, 3,...). \tag{14}$$

The scalar function $V(c, U)$ will be called a *utility aggregator*. We might interpret U_{t+1} in (14) as specifying an aggregate utility of a future consumption stream ${}_{t+1}C$, were it to be initiated immediately.

Stationary utility function (13) is ordinal in the sense that all its economically meaningful properties are invariant under any monotonic transformation of the utility scale. We assume that $U({}_1C)$ is strictly quasi-concave on the set of feasible consumption sequences, i.e.,

$$\begin{gathered} U(\theta({}_1C^1) + (1 - \theta)({}_1C^2)) > \min[U({}_1C^1), U({}_1C^2)] \\ \text{for any} \quad 0 < \theta < 1 \quad \text{and} \quad {}_1C^1 \neq {}_1C^2. \end{gathered} \tag{15}$$

It should be noted, however, that condition (15) does not necessarily imply quasi-concavity of the utility aggregator $V(c, U)$ with respect to c and U. It can be easily seen that, while quasi-concavity of the original stationary utility function is preserved by any nonlinear monotonic transformation, that of the aggregator is not.

We assume that the first derivatives of $V(c, U)$ with respect to c and U are both positive for $c > 0$ and possible range of U, i.e.,

$$\begin{aligned} \partial V(c, U)/\partial c &= V_c(c, U) > 0 \qquad \text{for } c > 0, \\ \partial V(c, U)/\partial c &= V_U(c, U) > 0 \qquad \text{for any possible } U. \end{aligned} \tag{16}$$

We also assume that $V(c, U)$ is continuous at $c = 0$ for any possible U, and that

$$\lim_{c \to 0} V_c(c, U) = +\infty \qquad \text{for any } U. \tag{17}$$

This condition is imposed so as to preclude corner solutions.

Note in passing that our stationary ordinal utility function includes

additive utility functions as a special case, since they can be written as

$$U_1 = \sum_{t=1}^{\infty} \beta^{t-1} u(c_t) = u(c_1) + \beta U_2, \tag{18}$$

where $(1/\beta) - 1 > 0$ is a systematic rate of time discount.

Now we have set up both the objective function and the technological constraints in our optimal growth model. Our planning authority wishes to maximize the intertemporal utility function (9) which summarizes our economy's time preference structure, subject to the stationary technological conditions (3) and to an initial capital endownment historically given. The problem confronting the planning authority can be formulated as follows:

$$\begin{gathered} \max U_1 = V(c_1, V(c_2, V(c_3, V((...))))), \\ \text{where } c_t + k_t \leqslant g(k_{t-1}), \qquad (t = 1, 2, 3,...), \\ 0 \leqslant k_0 \leqslant \bar{k}_0 \leqslant \bar{k}, \qquad \text{where } \bar{k}_0 \text{ is given historically,} \end{gathered}$$

where $\bar{k}_0$ is an initial capital stock, which can be assumed to be smaller than or equal to $\bar{k}$ without loss of generality.

III. Fisherian Intertemporal Equilibrium Condition

First of all, as the basis for our subsequent investigation, we can establish the following:

Existence and Uniqueness Theorem (Beals–Koopmans [1]). *For any initial capital stock $0 \leqslant \bar{k}_0 \leqslant \bar{k}$, there exists a unique optimal program represented by a consumption sequence ${}_1C$ and a capital sequence ${}_0K$.*

A sketch of the proof due to Beals and Koopmans is given in the footnote below.[8] Therefore, the remaining task for our planning authority is simply to characterize the nature of the optimal program whose existence and uniqueness have been guaranteed.

Let us define the Lagrangean function as

$$L = U({}_1C) - \sum_{t=1}^{\infty} a_t\{c_t + k_t - g(k_{t-1})\}, \tag{19}$$

[8] (Proof) Let Z be the set of feasible consumption sequences ${}_1C$. From the assumed properties of the concave function $g(k)$, (5)–(8), it can be shown that Z forms a bounded, strictly convex closed set and is compact with respect to the product topology. Hence, the continuous, strictly quasi-concave function $U({}_1C)$ attains a maximum at a unique consumption sequence in Z. Q.E.D.

where a_t is a Lagrangian multiplier associated with a technological constraint (3) in period t. The first-order conditions for this problem are

$$\partial U({}_1C)/\partial c_t - a_t = 0, \tag{20}$$

$$a_{t+1} g'(k_t) - a_t = 0, \tag{21}$$

$$c_t + k_t - g(k_{t-1}) = 0, \qquad \text{for} \quad t = 1, 2, 3, \ldots, \tag{22}$$

and

$$k_0 = \bar{k}_0 . \tag{23}$$

By (7) and (17) we could preclude corner solutions.

Then, from (20) and (21), we obtain the following equation

$$\frac{\partial U_1/\partial c_t}{\partial U_1/\partial c_{t+1}} = \frac{V_c(c_t, U_{t+1})}{V_U(c_t, U_{t+1})\, V_c(c_{t+1}, U_{t+2})} = \frac{a_t}{a_{t+1}} = g'(k_t). \tag{24}$$

This equation can be looked at as the well-known Irving Fisher proposition for intertemporal equilibrium.[9] It asserts that an optimal program must equalize a marginal rate of return over cost with a marginal rate of time preference in each period. In our model, the former is defined as the marginal rate of transformation between c_t and c_{t+1} keeping the other periods' consumptions constant, i.e.,

$$\partial c_{t+1}/\partial c_t - 1 = \{F_K(k_t, 1) - (n + \delta)\}/(1 + n) = g'(k_t) - 1. \tag{25}$$

And the latter might be defined as

$$\frac{\partial U_1/\partial c_t}{\partial U_1/\partial c_{t+1}} - 1 = \frac{V_c(c_t, U_{t+1})}{V_U(c_t, U_{t+1})\, V_c(c_{t+1}, U_{t+2})} - 1. \tag{26}$$

In the case of additive utility functions, the Fisherian condition can be simplified as

$$[u'(c_t)/\beta u'(c_{t+1})] - 1 = g'(k_t) - 1, \tag{27}$$

which depends only on c_t, c_{t+1} and k_t. This is nothing but the discrete time version of the Ramsey–Keynes condition for optimal saving.[10] In general, however, the marginal rate of time preference depends also on a perpetual consumption sequence ${}_{t+1}C$ and, in consequence, the Fisherian condition becomes an infinite-order difference equation. Although a direct

[9] See Fisher [4].

[10] See Ramsey [12].

attack on this infinite-order difference equation may be possible, we, as mortal beings, shall choose an indirect and roundabout method for analyzing this formidable problem.

IV. The Maximum Welfare Function and the Reduced Utility Function

In this section, two functions—*the maximum welfare function* and *the reduced utility function*—are introduced. These two functions will be the key to our optimal growth theory.

Let us first define the maximum welfare function, to be denoted by $W(k)$, as the maximal value of intertemporal utility U_1 attainable from an initial capital stock $k_0 = k$, i.e.,

$$W(k_0) = \max_{\substack{\{c_t \geqslant 0\} \\ \{k_t \geqslant 0\}}} U({}_1C) = V(c_1, V(c_2, V(c_3, \ldots))), \tag{28}$$

$$\text{where } c_t + k_t = g(k_{t-1}) \qquad (t = 1, 2, 3, \ldots),$$

$$\text{and } 0 \leqslant k_0 \leqslant \bar{k}, \qquad \text{where } k_0 \text{ is given.}$$

Since the existence and uniqueness of an optimal program have been already established, we can automatically assure the existence and uniqueness of this function for $0 \leqslant k_0 \leqslant \bar{k}$.

Our *principle of dynamic consistency* is the following proposition: An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.[11] It is obvious that this intuitive principle is not merely true for the usual dynamic programming models whose objective functions are additive functions, but also true for the more general case in which the objective function is recursive and satisfies only Koopmans' first three postulates (P-1)–(P-3). We shall call this generalized principle the principle of dynamic consistency, for it maintains that any optimal growth program must be dynamically consistent when both the objective function and the side conditions are recursive.

Now, according to this principle, if a consumption sequence ${}_1C$ and an associated capital sequence ${}_0K$ represent the unique optimal program for a given initial capital k_0, then truncated consumption and capital sequences, ${}_{t+1}C$ and ${}_tK$, must constitute the unique optimal program whose initial capital equals k_t. Consequently, by applying this principle to the

[11] This proposition is quoted from Bellman [2].

definition of the maximum welfare function (28), we obtain the following functional equation:

$$W(k_{t+1}) = \max U({}_tC) = \max V(c_t, U({}_{t+1}C)) = \max V(c_t, \max U({}_{t+1}C))$$
$$= \max_{\substack{c_t \geq 0 \\ k_t \geq 0}} V(c_t, W(k_t)) \quad \text{where } c_t + k_t = g(k_{t-1}),$$
$$(t = 1, 2, 3, \ldots). \tag{29}$$

At this moment it is convenient to define a new function:

$$\Phi(c, k) = V(c, W(k)). \tag{30}$$

We shall call this the reduced utility function. This function can be regarded as specifying the economy's single-period preference between consumption c during a period and capital stock k at the end of that period. Since $W(k_t)$ evaluates the future consumption stream ${}_{t+1}C$ optimal for an initial capital stock k_t transferring to the next period, we can also interpret $\Phi(c_t, k_t)$ as summarizing the society's subjective trade-off between an immediate consumption and perpetual consumption streams in the future.

Since the functional Eq. (29) is a succession of single-period constrained maximization problems, we can easily obtain necessary conditions for an optimal accumulation problem by solving them iteratively. But before doing that, we must examine the properties of the two functions, $W(k)$ and $\Phi(c, k)$.

First, it is easy to show that $W(k)$ is increasing in k. To see this, let ${}_1C$ and ${}_0K$ represent the optimal program for an initial capital $0 \leqslant k_0 < \bar{k}$. Then for a larger initial capital k_0' ($k_0 < k_0' \leqslant \bar{k}$), a consumption sequence ${}_1C' = (c_1', {}_2C)$ such that $c_1' > c_1$ is feasible for k_0' by consuming an extra output $g(k_0') - g(k_0)$ in the first period and pursuing the same capital sequence ${}_1K$ from the next period as the compared program. Hence we get the following relation,

$$U({}_1C') = V(c_1', U({}_2C)) > V(c_1, U({}_2C)) = U({}_1C) = W(k_0).$$

But by the definition of $W(k)$, we have

$$W(k_0') \geqslant U({}_1C') > W(k_0) \quad \text{for } \bar{k} \geqslant k_0' > k_0 \geqslant 0, \tag{31}$$

which proves that $W(k)$ is increasing in k. Furthermore, if we assume

differentiability of $W(k)$, we can evaluate the first derivative by applying the well-known Envelope theorem.[12]

$$W'(k_0) = \partial L/\partial k_0 = a_1 g'(k_0) \qquad \text{(by (19))}$$
$$= g'(k_0) \cdot \{\partial V(c_1, U_2)/\partial c_1\}| \text{ (at the optimum)} \quad \text{for } 0 < k_0 < \bar{k}. \tag{32}$$

We cannot, however, determine the sign of $W''(k)$, if it exists, because of ordinality of the stationary utility function. In other words, convexity/concavity of the maximum welfare function is not preserved by all monotonic transformations of the original utility scale.

On the other hand, we can prove the following important proposition concerning the nature of the reduced utility function. That is, $\Phi(c, k)$ is strictly quasi-concave with respect to c and k, i.e.,

$$\Phi(\theta c^1 + (1-\theta)\, c^2, \theta k^1 + (1-\theta)\, k^2) > \min[\Phi(c^1, k^1), \Phi(c^2, k^2)]$$
$$\text{for any} \quad (c^1, k^1) \neq (c^2, k^2) \quad \text{and} \quad 0 < \theta < 1. \tag{33}$$

This property is clearly an ordinal one and is invariant under any order preserving transformation of the utility scale.

To prove (33), let us represent two accumulation programs from $t = 1$, respectively optimal for $0 \leqslant k_1^i \leqslant \bar{k}$, $(i = 1, 2)$, by ${}_2C^i$ and ${}_1K^i$. It follows from concavity of $g(k)$ that the set of feasible consumption sequence producible from a given initial capital stock forms a closed and convex set. So a consumption sequence $\theta({}_2C^1) + (1-\theta)({}_2C^2)$, a convex combination of the two consumption sequences optimal for k_1^1 and k_1^2, is feasible for an initial capital stock $\theta k_1^1 + (1-\theta)\, k_1^2$. Then by the definition of $W(k)$, we obtain

$$W(\theta k_1^1 + (1-\theta)\, k_1^2) \geqslant U(\theta({}_2C^1) + (1-\theta)({}_2C^2)).$$

Hence, by (12) we get

$$\begin{aligned}
&\Phi(\theta c_1^1 + (1-\theta)\, c_1^2, \theta k_1^1 + (1-\theta)\, k_1^2) \\
&\quad = V(\theta c_1^1 + (1-\theta)\, c_1^2, W(\theta k_1^1 + (1-\theta)\, k_1^2)) \\
&\quad \geqslant V(\theta c_1^1 + (1-\theta)\, c_1^2, U(\theta({}_2C^1) + (1-\theta)({}_2C^2)) \\
&\quad = U(\theta({}_1C^1) + (1-\theta)({}_1C^2)).
\end{aligned}$$

However, by strict quasi-concavity of the stationary ordinary utility

[12] See, for example, Samuelson [13].

function, the right-hand side of the above inequality is strictly greater than

$$\begin{aligned}\min[U({}_1C^1), U({}_1C^2)] &= \min[V(c_1^1, U({}_2C^1)), V(c_1^2, U({}_2C^2))] \\ &= \min[V(c_1^1, W(k_1^1)), V(c_1^2, W(k_1^2))].\end{aligned}$$

The last equality follows from the supposition that ${}_2C^i$ is optimal for k_1^i. This immediately leads to the desired inequality:

$$\Phi(\theta c_1^1 + (1-\theta)\, c_1^2, \theta k_1^1 + (1-\theta)\, k_1^2) > \min[\Phi(c_1^1, k_1^1), \Phi(c_1^2, k_1^2)].$$

If we assume twice-differentiability of $\Phi(c, k)$, the strict qasi-concavity of $\Phi(c, k)$ is equivalent to negative-definiteness of the bordered Hessian matrix H defined by

$$H = \begin{bmatrix} 0 & \Phi_c & \Phi_k \\ \Phi_c & \Phi_{cc} & \Phi_{ck} \\ \Phi_k & \Phi_{kc} & \Phi_{kk} \end{bmatrix} = \begin{bmatrix} 0 & V_c & V_U W' \\ V_c & V_{cc} & V_{cU} W' \\ V_U W' & V_{Uc} W' & V_{UU} W'^2 + V_U W'' \end{bmatrix}, \quad (34)$$

where $\Phi_x = \partial\Phi/\partial x$, $\Phi_{xy} = \partial^2\Phi/\partial x\, \partial y$, $V_{xy} = \partial^2 V/\partial x\, \partial y$ and $W'' = d^2W/dk^2$.

In the appendix we shall show that $\Phi(c, k)$ is continuous for $c \geqslant 0$ and $0 \leqslant k \leqslant \bar{k}$. Furthermore, we can show the following conditions if differentiability is assumed:

$$\Phi_c > 0 \quad \text{for} \quad c > 0 \quad \text{and} \quad \Phi_k > 0 \quad \text{for} \quad 0 < k \leqslant \bar{k}. \quad (35)$$

$$\lim_{c\to 0} \Phi_c = +\infty \quad \text{and} \quad \lim_{k\to 0} \Phi_k = +\infty. \quad (36)$$

So far we have discussed the qualitative properties of $W(k)$ and $\Phi(c, k)$ as if we could take these functions as given. However, *a priori* we can regard only the technological function $g(k)$ and the utility aggregator $V(c, U)$ as given and the functional forms of $W(k)$ and $\Phi(c, k)$ must be determined by these two given functions. To fill this logical gap and make our optimal growth theory self-contained, we shall supply in the appendix a computational method which can determine $W(k)$ and $\Phi(c, k)$ from $g(k)$ and $V(c, U)$ only by successive functional approximations. Therefore, in the following sections in which we shall construct optimal programs, we can legitimately suppose the functional forms of both the maximum welfare function and the reduced utility function as given.

V. Single-period Fisherian Diagram

In the following three sections, we shall illustrate the generation of optimal programs by simple two-dimensional diagrams. Our geometrical approach might be considered as an extension of Irving Fisher's graphical

exposition of two-period models of capital theory to the more general capital accumulation models involving infinite periods.[13]

We have already remarked that by virtue of the principle of dynamic consistency, an optimal program must satisfy the following functional equation for $t \geqslant 1$.

$$W(k_{t-1}) = \max_{\substack{c_t \geqslant 0 \\ k_t \geqslant 0}} \Phi(c_t, k_t) \quad (37)$$

where

$$c_t + k_t = g(k_{t-1}). \quad (38)$$

Our starting point is to regard this functional equation as a succession of single-period optimization problems—to maximize the reduced utility function subject to the technological constraints. A side constraint (38) represents a single-period production possibility frontier producible from a given initial capital stock k_{t-1} inherited from the last period. Since we have assumed one-sector production model, this frontier is a straight line whose slope is minus one.[14]

Figure 2 explains how to construct a single-period production possibility frontier from a given k_{t-1} in our modified Fisherian diagram. The horizontal axis measures capital stock k and the vertical axis measures consumption c. Curve OBE represents the schedule of maximum amount of consumption: max $c_t = g(k_{t-1})$ which would be feasible if the economy decided not to invest any capital stock for production purpose at all. We draw a perpendicular line from the point of initial capital stock on the horizontal axis, say A, and then find its intersection, say B, with the maximum consumption schedule. The length AB equals $g(k_{t-1})$. We can easily find a point C on the vertical axis whose height is equal to AB. Finally, we draw a straight line CD from C whose slope is $-45°$. This CD

[13] Irving Fisher himself was rather pessimistic about the possibility of dynamization of his graphical exposition of two-period consumption programs. (See p. 287 in Fisher [4].) This view was endorsed by Westfield [18]. However, contrary to their pessimistic view, our geometrical analysis in the subsequent sections may give a partial justification for Leontief's attempt to visualize consumption paths involving many periods in the Fisherian two-dimensional diagram. (See [9] and [10].) It should be noted that our modified Fisherian diagram represents future consumption sequences by capital stock at the end of a period using the maximum welfare function.

[14] We can easily extend our analysis to the more general two-sector production model in which the single-period production possibility frontier is concave towards the origin.

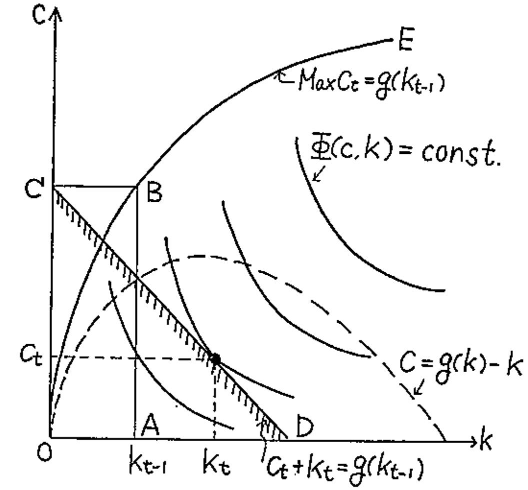

FIG. 2. Determination of a single-period optimal point in a Fisherian diagram.

line is nothing but the single-period production possibility frontier.[15] It should be noted that the intersections of CD with AB traces out the concave curve $c = g(k) - k$.

On the other hand, the objective function (37) in the above single-period problem is the reduced utility function which can be looked at as specifying the economy's single-period preference between c_t and k_t. We can draw a family of indifference curves of the utility function in the same way as in the standard static utility theory. By strict quasi-concavity of the reduced utility function, all the indifference contours are strictly convex towards the origin of our Fisherian diagram. Since the form of the reduced utility function is stationary over time, these indifference contours are also invariant over time.

As Fig. 2 indicates, the optimal point (k_t, c_t) in our single-period optimization problem is determined by a tangential point between the production possibility frontier and one of the indifference curves of the reduced utility function. Since the indifference contours are strictly convex this single-period optimum is unique.

Instead we can determine the single-period optimum by use of calculus; by differentiating a Lagrangean function $\Phi(c_t, k_t) + \lambda\{g(k_{t-1}) - c_t - k_t\}$

[15] Here, we implicitly assume that capital stock is completely malleable and can be instantaneously melted into any kind of consumption goods. If we cannot eat and/or wear second-hand machines, this frontier becomes horizontal at the level of $c = F(k, 1)/(1 + n)$.

with respect to c_t, k_t and the Lagrangean multiplier λ, equating the derivatives to zero and eliminating λ, we obtain

$$-\Phi_c(c_t, k_t)/\Phi_k(c_t, k_t) = -1. \tag{40}$$

$$c_t + k_t = g(k_{t-1}). \tag{41}$$

By (36), corner solutions could be excluded. That these two conditions uniquely determine a single-period maximum is assured by continuity and strict quasi-concavity of the reduced utility function $\Phi(c, k)$.

Equation (40) is easy to interpret. It asserts that as a necessary condition for an accumulation program to be optimal a marginal rate of substitution between c_t and k_t in terms of the reduced utility function, $-\Phi_c/\Phi_k$, must be equal to a marginal rate of technological transformation between c_t and k_t which is minus one in our one-sector production model.

Furthermore, if we substitute the envelope relation (32) into (40), we obtain the following equation for $0 < k_t < \bar{k}$:

$$[V_c(c_t, W(k_t))/V_U(c_t, W(k_t))\ V_c(c_{t+1}, W(k_{t+1}))] - 1 = g'(k_t) - 1. \tag{42}$$

A moment reflection makes us realize that (42) is again nothing but the Fisherian intertemporal equilibrium condition discussed in Section III. However, different from the previous Fisherian condition (24), aggregate future utilities in this new condition are evaluated by the maximum welfare function.

VI. Dynamic Fisherian Diagram

Now let us make our single-period Fisherian diagram dynamic.

Suppose we have already determined the optimal combination of c_{t-1} and k_{t-1} that satisfies the tangential condition (40) in period $t - 1$. We can construct a new single-period production possibility frontier for period t on the base of k_{t-1}. The position of this new frontier is determined by the capital accumulation Eq. (41). This equation can be rewritten as

$$k_t - k_{t-1} = \{g(k_{t-1}) - k_{t-1}\} - c_t. \tag{43}$$

Therefore, the net saving is positive, zero or negative according as $c_t - \{g(k_{t-1}) - k_{t-1}\}$ is negative, zero or positive, respectively. The schedule $c = g(k) - k$ will be called *the steady state schedule*, for no net accumulation and decumulation will occur on this schedule. This schedule is depicted as an inverted U-shaped curve in Fig. 3.

When the optimal point in $t-1$ happened to be on the steady state schedule, the production possibility frontier will not shift thereafter and the economy will remain on that point forever. When the optimal point in $t-1$ lies in the lower region of this schedule, our economy accumulates capital stock and, in consequence, the new production possibility frontier

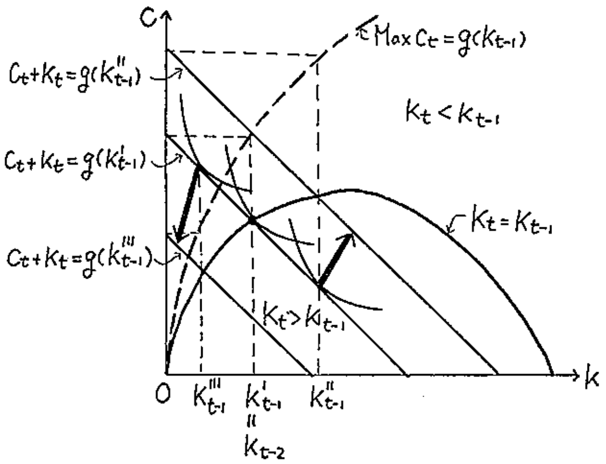

FIG. 3. Accumulation and decumulation of capital stock.

will be expanded. Also when the previous optimal point is in the upper region of the steady state schedule, the economy decumulates capital and the new frontier will shrink towards the origin (see Fig. 3). We might call the lower part of the steady state schedule the region of capital accumulation and the upper part the region of capital decumulation.

In period t, an optimal point is determined on the new production possibility frontier so as to satisfy the tangential condition (40) for single-period optimum. The transit from (k_{t-1}, c_{t-1}) to (k_t, c_t) can be clearly identified with an *Engel curve* or an *income-consumption line* of our reduced utility function, traced out by shifting the economy's budget line $=$ the single-period production possibility frontier in a parallel manner. Every schoolboy knows that the direction of an Engel curve hinges upon normality/neutrality/inferiority of commodities. In our reduced utility system, for instance, when both c_t and k_t are normal goods, the Engel curve must have a positive slope in the Fisherian diagram.

Let us define income effects on capital stock and immediate consumption in period t, respectively, by

$$I(k_t) = \left.\frac{\Delta k_t}{\Delta y_t}\right|_{(40)\text{ and }(41)}, \tag{45}$$

$$I(c_t) = \left.\frac{\Delta c_t}{\Delta y_t}\right|_{(40)\text{ and }(41)}, \tag{46}$$

where $y_t = g(k_{t-1})$ and Δ is the difference operator defined by $\Delta x_t = x_t - x_{t-1}$. Capital stock (consumption) is said to be normal, neutral and inferior in period t if $I(k_t)$ $(I(c_t))$ is positive, zero and negative, respectively. Note that by the budget Eq. (41) these two income effects must satisfy the following equation:

$$I(k_t) + I(c_t) = 1. \tag{47}$$

If we assume twice-differentiability of $\Phi(c, k)$, their first-order approximations can be calculated by differentiating the single-period optimality conditions (40) and (41) totally with respect to $y_t = g(k_{t-1})$ and solving the resulting linear equations. Thus we have

$$I(k) \cong D_k \Phi_c / D, \tag{48}$$

$$I(c) \cong D_c \Phi_c / D, \tag{49}$$

where

$$D = \| H \| \tag{50}$$

is the determinant of the bordered Hessian matrix of $\Phi(c, k)$;

$$D_k = \begin{Vmatrix} \Phi_c & \Phi_{cc} \\ \Phi_k & \Phi_{kc} \end{Vmatrix} = \begin{Vmatrix} V_c & V_{cc} \\ V_U W' & V_{Uc} W, \end{Vmatrix} = W' \begin{Vmatrix} V_c & V_{cc} \\ V_U & V_{Uc} \end{Vmatrix}. \tag{51}$$

is the cofactor of the element in the first row and the second column of H; and

$$D_c = -\begin{Vmatrix} \Phi_c & \Phi_{ck} \\ \Phi_k & \Phi_{kk} \end{Vmatrix} = -\begin{Vmatrix} V_c & V_{cU} W' \\ V_U W' & V_{UU} W'^2 + V_U W'' \end{Vmatrix}$$

$$= -W'^2 \begin{Vmatrix} V_c & V_{cU} \\ V_U & V_{UU} \end{Vmatrix} - V_c V_U W'' \tag{52}$$

is the cofactor of the element in the first row and the third column of H. Note in passing that the sign of $I(k)$ is dependent only on the sign of

$\left\| \begin{smallmatrix} V_c & V_{cc} \\ V_U & V_{cU} \end{smallmatrix} \right\|$ and might be inferred solely from the utility aggregator $V(c, U)$ without actually computing the functional form of $\Phi(c, k)$. However, by (52) such a short-cut inference of the sign of $I(c)$ seems impossible.

Now, since $\Delta y_t/\Delta k_{t-1} \cong g'(k_{t-1}) > 0$ by (6), we can deduce the following relations between the income effects and the monotonicity of capital and consumption sequences from (45) and (46):

$$I(k_t)\{\gtreqless\} 0 \Leftrightarrow \Delta k_t/\Delta k_{t-1}\{\gtreqless\} 0, \tag{53}$$

$$I(c_t)\{\gtreqless\} 0 \Leftrightarrow \Delta c_t/\Delta k_{t-1}\{\gtreqless\} 0. \tag{54}$$

Therefore, in order to determine the direction of the Engel curve, we must distinguish at least five cases according to the possible combinations of the signs of $I(k_t)$ and $I(c_t)$. [By (47), either $I(k_t)$ or $I(c_t)$ or both must be positive.]

(i) *$I(k_t) > 0$ and $I(c_t) > 0$*

When both capital and consumption are normal in period t, we have by (53) and (54) one of the following two cases if $k_{t-1} \neq k_{t-2}$:

$$\begin{aligned} & k_t > k_{t-1} \quad \text{and} \quad c_t > c_{t-1} \quad \text{when} \quad k_{t-1} > k_{t-2}, \\ \text{and} \quad & k_t < k_{t-1} \quad \text{and} \quad c_t < c_{t-1} \quad \text{when} \quad k_{t-1} < k_{t-2}. \end{aligned} \tag{55}$$

Therefore, in our Fisherian diagram the Engel curve is positively sloped, and both k_t and c_t are increased (decreased) as the single-period production possibility frontier is expanded (contracted).

(ii) *$I(k_t) > 0$ and $I(c_t) = 0$*

When consumption becomes neutral in period t, the level of consumption in this period must be the same as in the previous period, for by (53) and (54) we have

$$\begin{aligned} & k_t > k_{t-1} \quad \text{and} \quad c_t = c_{t-1} \quad \text{when} \quad k_{t-1} > k_{t-2}, \\ \text{and} \quad & k_t < k_{t-1} \quad \text{and} \quad c_t = c_{t-1} \quad \text{when} \quad k_{t-1} < k_{t-2}. \end{aligned} \tag{56}$$

The Engel curve becomes a horizontal line and any expansion (contraction) of the single-period production possibility frontier is entirely absorbed by an increase (decrease) in capital stock, keeping the level of consumption constant.

(iii) *$I(k_t) > 0$ and $I(c_t) < 0$*

When consumption is inferior in period t, (53) and (54) imply the following relations:

$$\begin{aligned} & k_t > k_{t-1} \quad \text{and} \quad c_t < c_{t-1} \quad \text{when} \quad k_{t-1} > k_{t-2}, \\ \text{and} \quad & k_t < k_{t-1} \quad \text{and} \quad c_t > c_{t-1} \quad \text{when} \quad k_{t-1} < k_{t-2}. \end{aligned} \tag{57}$$

In our Fisherian diagram, the Engel curve bends forwards, and the economy decreases (increases) its consumption level even if the single-period production possibility set becomes larger (smaller) than in the previous period.

In the three cases above in which capital stock is a normal good for the economy, the first relation in (53) implies that the motion of capital sequences is monotone in time. Furthermore, the monotonicity of capital sequences leads to

$$k_{t-1}\{\gtreqless\} k_{t-2} \quad \text{implies} \quad c_t\{\lesseqgtr\} g(k_{t-1}) - k_{t-1}, \tag{58}$$

because $c_t = g(k_{t-1}) - k_t$. Therefore, in these cases, we observe that the new single-period optimal point (k_t, c_t) does not exceed the steady-state schedule but remains in the same capital accumulation or decumulation region as the previous period's optimal point (k_{t-1}, c_{t-1}).

(iv) *$I(k_t) = 0$ and $I(c_t) > 0$*

When capital stock happens to be neutral in period t, we have

$$\begin{aligned} & k_t = k_{t-1} \quad \text{and} \quad c_t > c_{t-1} \quad \text{when} \quad k_{t-1} > k_{t-2}, \\ \text{and} \quad & k_t = k_{t-1} \quad \text{and} \quad c_t < c_{t-1} \quad \text{when} \quad k_{t-1} < k_{t-2}. \end{aligned} \tag{59}$$

The constancy of capital stocks as indicated above, in turn, implies

$$c_t = g(k_{t-1}) - k_t = g(k_{t-1}) - k_{t-1}. \tag{60}$$

Hence, in this neutral capital case, we observe that the Engel curve becomes vertical and the economy jumps to a point on the steady state schedule in that period. Geometrically, this interesting result is confirmed by the fact that the single-period production possibility frontier in period t intersects the steady state schedule at the point whose abscissa equals k_{t-1}.

(v) $I(k_t) < 0$ *and* $I(c_t) > 0$

Finally, when capital stock becomes inferior in period t, we find that the Engel curve becomes backward-bending in our Fisherian diagram, for we obtain from (53) and (54) one of the following two cases:

$$\begin{array}{lllll} & k_t < k_{t-1} & \text{and} & c_t > c_{t-1} & \text{when} \quad k_{t-1} > k_{t-2}, \\ \text{and} & & & & \\ & k_t > k_{t-1} & \text{and} & c_t < c_{t-1} & \text{when} \quad k_{t-1} < k_{t-2}. \end{array} \tag{61}$$

Moreover, from this we have the following relations:

$$k_{t-1}\{\gtreqless\}\, k_{t-2} \quad \text{implies} \quad c_t\{\gtreqless\}\, g(k_{t-1}) - k_{t-1}. \tag{62}$$

Therefore, if (k_{t-1}, c_{t-1}) was below (above) the steady state schedule, i.e., if it is in the region of capital accumulation (decumulation), the new optimal point (k_t, c_t) overshoots this schedule and necessarily enters into the region of capital decumulation (accumulation). The motion of capital sequences ceases to be monotone in this inferior capital case.

The above rather tedious discussions are summarized in Fig. 4 only for the case where capital was accumulated from period $t-2$ to $t-1$.

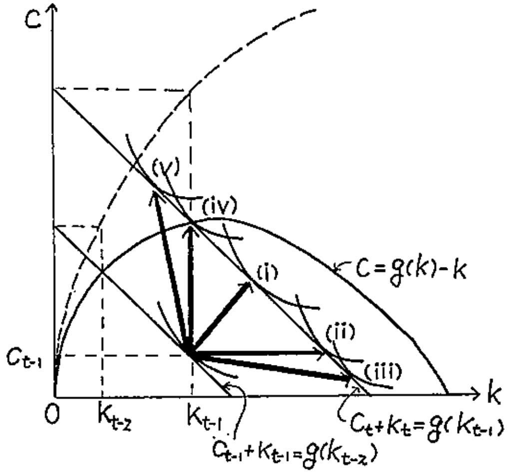

FIG. 4. Directions of Engel curves of the reduced utility function.

If the piecemeal Engel curves constructed by our Fisherian diagrammatic technique are successively connected from $t = 0$ to $t = +\infty$, we can determine the whole sequence of k_t and c_t that satisfy our functional Eq. (37) for a given initial capital stock $0 \leqslant k_0 \leqslant \bar{k}$. Continuity and strict quasi-concavity of the reduced utility function assure the existence and uniqueness of such sequence.

Now it is clear that this succession of piecemeal Engel curves always moves along a curve which is traced out by the combinations of consumption and capital that satisfy the single-period tangential condition (40). We shall call this curve *the dynamic Engel curve.*

However, a question still remains: Is the program thus constructed along the dynamic Engel curve indeed optimal? To this question, we can give an affirmative answer. The reasoning goes as follows. First, the existence and uniqueness theorem assures that there exists exactly one optimal program for any initial capital $0 \leqslant k_0 \leqslant \bar{k}$. Secondly, the principle of dynamic consistency asserts that this optimal program must satisfy the functional Eq. (37) for all t. And, thirdly, continuity and strict quasi-concavity of $\Phi(c, k)$ guarantees one-to-one relationship between (37) and the program constructed in our Fisherian diagram. As a logical consequence, we can conclude that the unique program determined by our Fisherian diagram must be optimal, *if* the functional form of the reduced utility function

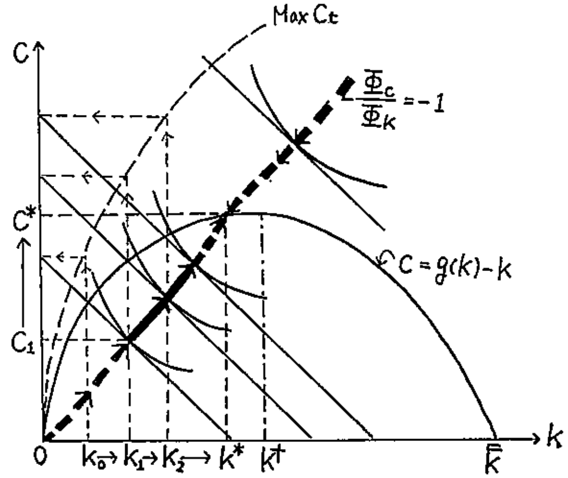

FIG. 5. Generation of an optimal program on the dynamic Engel curve.

could be taken as given. Fortunately, the last qualification can be eliminated, since we can, in fact, determine $\Phi(c, k)$ from the given functions $g(k)$ and $V(c, U)$ only by applying a method of successive approximations to be developed in the appendix. Figure 5 illustrates the generation of an optimal program on the dynamic Engel curve, although it by no means exhausts possible configurations.

VII. Asymptotic Behaviors of Optimal Programs

It now remains to give a detailed account of possible asymptotic behaviors of optimal paths. Since they always move along the dynamic Engel curve defined by (40), our investigation of their asymptotic behaviors can be reduced to the simple analysis of this curve's properties.

An intersection of the dynamic Engel curve and the steady state schedule will be called a *generalized turnpike*. Analytically, it is defined as a root, (k^*, c^*), of the equations

$$-\Phi_c(c^*, k^*)/\Phi_k(c^*, k^*) = -1,$$
$$c^* + k^* = g(k^*), \qquad 0 \leqslant k^* \leqslant \bar{k}. \qquad 0 \leqslant c^*. \tag{63}$$

If (63) is substituted into the Fisherian equilibrium condition (42), we obtain the following equation for $0 < k^* < \bar{k}$:

$$g'(k^*) = 1/V_U(c^*, W(k^*)). \tag{64}$$

However, by differentiating the identity $U(C^*) = V(c^*, U(C^*))$, where C^* denotes a steady state consumption sequence $(c^*, c^*, c^*, ...)$, we can easily show that

$$0 < V_U(c^*, U(C^*)) < 1. \tag{65}$$

Hence, the left-hand side of (64) must be greater than unity, yielding

$$g'(k^*) > 1 = g'(k^\dagger). \tag{66}$$

Therefore, by (8) we can say that an optimal steady state capital stock k^*, if any, must be smaller than the golden rule capital stock $k^\dagger$.

$$0 \leqslant k^* < k^\dagger < \bar{k}. \tag{67}$$

The existence of a generalized turnpike is trivially assured by the fact that the origin of the Fisherian diagram, $k^* = c^* = 0$, satisfies (63). (Note that this zero-consumption, zero-capital program is obviously optimal for zero initial capital stock.) Furthermore, in a special case where the intertemporal utility function is an additive function like (17), the right-hand side of (64) becomes constant and consequently by (7) we can guarantee the existence and uniqueness of a nonzero generalized turnpike other than the trivial one. However, in the general case in which $1/V_U$ is not constant, it cannot be assured that there exists a nontrivial generalized turnpike. Moreover, even if we could find a nontrivial turnpike, the possibility of multiple turnpikes cannot be excluded.[16]

Let us put together all the previous results and describe possible asymptotic behaviors of optimal programs in our two-dimensional Fisherian diagrams.

(i) If both capital stock and consumption are normal in all periods, the dynamic Engel curve is positively sloped everywhere. In this case, from (55) both the sequences of optimal capitals and optimal consumptions are monotone in time. But even in this well-behaved case, we can assure neither existence nor uniqueness of a nontrivial generalized turnpike. Figure 6 explains what are essentially all possible types of generalized turnpikes in this case.

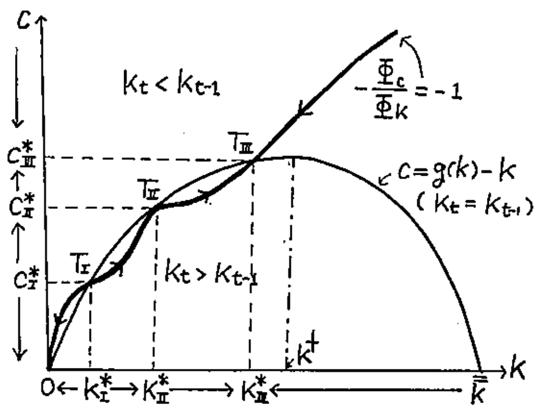

Fig. 6. Asymptotic behaviors of optimal programs when both consumption and capital are normal.

Point T_{III}, (k^*_{III}, c^*_{III}), represents an asymptotically stable turnpike. By asymptotic stability we mean that for every program optimal for k_0 sufficiently close to k^*_{III}, $\lim_{t\to\infty} k_t = k^*_{III}$ and $\lim_{t\to\infty} c_t = c^*_{III}$. At T_{III} the dynamic Engel curve crosses the steady state schedule from below. So it is easy to see that for k_{t-1} sufficiently close to k^*_{III}

$$k_{t-1} > k^*_{III} \quad \text{implies} \quad k_t < k_{t-1},$$
and
$$k_{t-1} < k^*_{III} \quad \text{implies} \quad k_t > k_{t-1}. \tag{68}$$

[16] Kurz [8] shows that there may exist multiple turnpikes if we introduce capital as an argument in the Ramsey type utility function. Note that our reduced utility $\Phi(c, k)$ contains capital as an argument, and that its formal structure is somewhat similar to his model.

Furthermore, since the dynamic Engel curve is positively sloped it follows that (68) can be strengthened as follows.

$$\begin{aligned} & k_{t-1} > k^*_{III} \quad \text{implies} \quad k^*_{III} < k_t < k_{t-1}, \\ \text{and} \quad & k_{t-1} < k^*_{III} \quad \text{implies} \quad k^*_{III} > k_t > k_{t-1}. \end{aligned} \tag{69}$$

Thus we have established not only the asymptotic stability of T_{III} but also the monotonic convergence of optimal programs to it. By the same token we can easily show that the trivial turnpike in this example is asymptotically stable from the right.

Point T_I, (k_I^*, c_I^*), at which the dynamic Engel curve crosses the steady state schedule from above, represents an asymptotically unstable generalized turnpike. It can be easily seen that every program optimal for k_0 sufficiently close but unequal to k_I^* monotonically diverges from T_1. Note that if k_0 happens to be equal to k_I^* the optimal porgram keeps staying on T_I until some random shock disturbs this knife-edge equilibrium.

Point (k^*_{II}, c^*_{II}) is an example of one-sided stable-unstable generalized turnpikes. In this example, where the dynamic Engel curve touches the steady state schedule from below, T_{II} is asymptotically stable from the left and asymptotically unstable from the right.

Let us summarize our Fig. 6; if $k_0 < k_I^*$, the optimal program leads only to forever diminishing consumption and capital stock and gradually converges to the situation of zero consumption; if $k_I^* < k_0 < k^*_{II}$, the optimal program monotonically converges to T_{II} by accumulating capital; if $k^*_{II} < k_0 \leqslant \bar{k}$, T_{III} serves as a long-run target of optimal programs; from $k^*_{II} < k_0 < k^*_{III}$ they approach T_{III} by accumulating capital and from $k^*_{III} < k_0 \leqslant \bar{k}$ they approach it by decumulating capital; finally if it so happens that k_0 equals one of $(0, k_I^*, k^*_{II}, k^*_{III})$ the optimal program will remain on that generalized turnpike forever.

(ii) & (iii) In the case where immediate consumption becomes inferior or neutral in some period, the dynamic Engel curve has a forward-bending or horizontal portion. From (56) and (57), the dynamic behavior of optimal consumption sequences ceases to be monotone, while that of optimal capital sequences remains monotone in time. As is shown in Fig. 7 a generalized turnpike around which consumption is inferior is necessarily an asymptotically unstable one. To see this we only have to recall (64) which implies that every generalized turnpike is located on the positively sloped part of the steady state schedule. This fact clearly excludes the possibility of the negatively sloped dynamic Engel curve intersecting the steady state schedule from below.

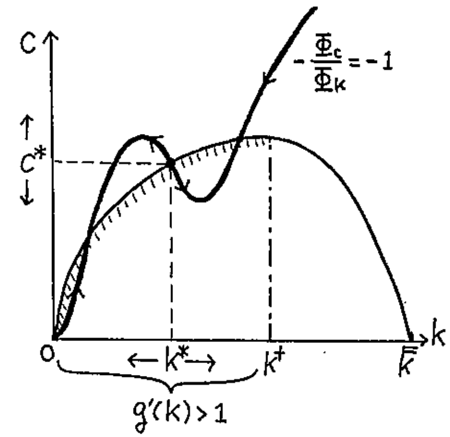

FIG. 7. Asymptotic behaviors of optimal programs when consumption is inferior.

Altenatively, this can be seen as follows. By (45) and (47), we get the following equation:

$$\Delta k_t/\Delta k_{t-1} = (\Delta k_t/\Delta y_t)(\Delta y_t/\Delta k_{t-1}) = (1 - I(c_t))\,\Delta g(k_{t-1})/\Delta k_{t-1}.$$

Since $I(c_t) < 0$ and $g'(k_{t-1}) > 1$ in the neighborhood of the generalized turnpike, the left-hand side of the above equation is greater than unity. Hence, we obtain an inequality for an optimal capital sequence sufficiently close but unequal to k^*:

$$| k_t - k_{t-1} | > | k_{t-1} - k_{t-2} | > 0,$$

which implies that a sequence $\{| k_t - k_{t-1} |\}$ is bounded away from zero for every $k_0 \neq k^*$, proving the asymptotic instability of k^*.

(iv) When capital stock becomes neutral in period t, the optimal program immediately jumps onto a point of the steady state schedule and will perpetuate the stationary movement henceforward. The generalized turnpike in this case is not an asymptotically stable one but a perfectly stable one in the sense that the optimal program returns to it instantaneously when it is displaced.

(v) In the case where capital becomes inferior in some period, the dynamic Engel curve bends backwards. It is easily seen by iterating the relations (61) and (62) that optimal programs in this inferior capital case must oscillate around the steady state schedule between the region of capital accumulation and the region of capital decumulation from then on. As is shown in Fig. 8, if inferiority of capital is relatively weak, the oscillation

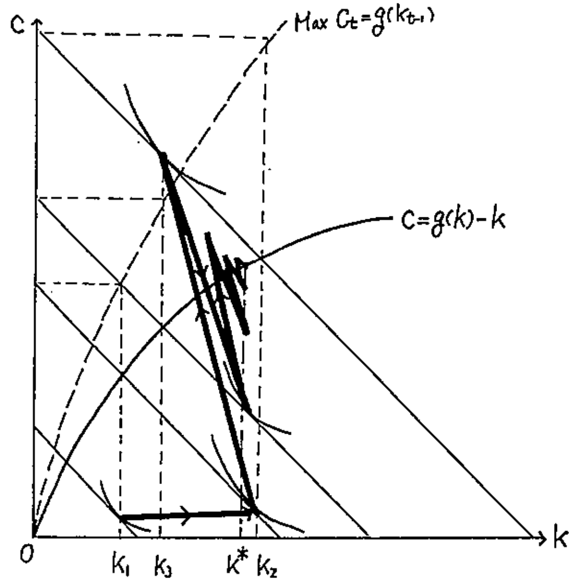

FIG. 8. Cyclical behaviors of optimal programs when capital is inferior.

will fade away and the optimal program will asymptotically converge to a generalized turnpike. If inferiority is strong, the optimal program will oscillatorily diverge from the generalized turnpike and will either approach another generalized turnpike or converge to a periodic motion (one of cobweb boxes as depicted in Fig. 9) around the generalized turnpike.

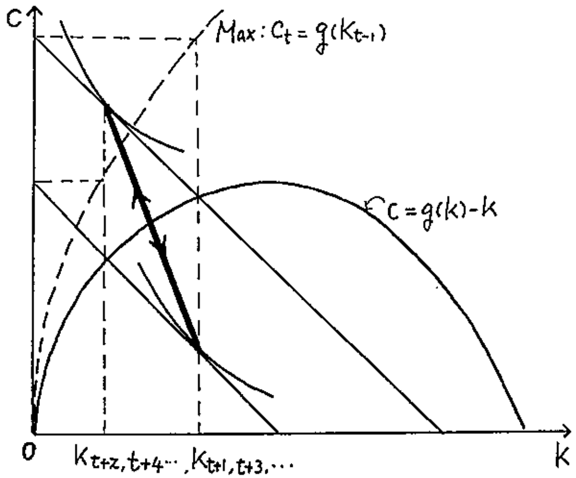

FIG. 9. A cobweb-cycle box of optimal programs when capital is strongly inferior.

These remarks on stability of oscillatory turnpikes can be confirmed by the following identity:

$$\Delta k_t/\Delta k_{t-2} = I(k_t)\, I(k_{t-1})(\Delta g(k_{t-1})/\Delta k_{t-1})(\Delta g(k_{t-2})/\Delta k_{t-2}). \tag{72}$$

This implies that if $0 < -I(k) < 1/g'(k)$ in the neighborhood of k^*, then we have an inequality

$$\Delta k_t/\Delta k_{t-2} < 1. \tag{73}$$

Hence, for k_0 close to k^* we have monotonically decreasing sequence $\{| k_t - k_{t-1} |\}$ for $t = 1, 3, 5,..., 2n + 1,...$. Since the dynamic Engel curve is negatively sloped, (61) implies one of the following:

$$k_0, k_2, ..., k_{2n}, ... < k^* < k_1, k_3, ..., k_{2n+1}, ...,$$

and

$$k_0, k_2, ..., k_{2n}, ... > k^* > k_1, k_3, ..., k_{2n+1}, ...$$

This, together with (73), shows that the generalized turnpike k^* is asymptotically stable. Similarly, if $-I(k) > 1/g'(k)$ in the neighborhood of k^*, we have from (72),

$$\Delta k_t/\Delta k_{t-2} > 1, \tag{73}$$

which shows that k^* cannot be asymptotically stable.

(vi) In general, the dynamic Engel curve may behave in a complicated way. There may coexist monotonically stable, monotonically unstable, one-sided stable-unstable, oscillatory stable and oscillatory unstable generalized turnpikes and perputually cyclical cobweb boxes in our optimal growth model.[17] But since the range of feasible capital stocks is bounded below by zero and above by $\bar{k}$, any optimal program must either approach one of the generalized turnpikes including the trivial one or approach a periodic motion of some period. In any case, the dynamic behavior of optimal programs in our general model is not so simple as in the Ramsey theory of optimal saving unless we impose some restrictions on the properties of the stationary ordinal utility function.

[17] Beals–Koopmans [1] covers the cases (i)–(iii) where optimal capital sequences can be shown to be monotone in time. (Optimal consumption sequences may not be monotone.) Since the proof of the existence and uniqueness theorem does not depend on the assumption of normal capital stock, oscillatory accumulation programs can be optimal, although we have not been able to construct an example yet.

APPENDIX: Computation of the Maximum Welfare Function and the Reduced Utility Function

In the text we developed the diagrammatic technique that can determine the unique optimal program for any initial capital stock on the condition that we already know the functional forms of $W(k)$ and $\Phi(c, k)$. However, *a priori* only the technological function $g(k)$ and the utility aggregator $V(c, U)$ were given. The purpose of this appendix is to supply a computational method which can determine $W(k)$ and $\Phi(c, k)$ solely from these given functions. Essentially, our computational method is a modification of successive functional approximations widely used in the theory of dynamic programming.[18]

Let us consider the sequence of functions $\{W_n(k_0)\}$, $n = 1, 2, 3, \ldots$, defined as follows:

$$W_1(k_0) = \max_{\{c \geq 0, \, k \geq 0\}} V(c, \underline{U}), \quad \text{where } c + k = g(k_0), \quad 0 \leq k_0 \leq \bar{k}, \tag{A-1}$$

and $\underline{U} = U(0, 0, 0, \ldots) > -\infty$ is the aggregate utility of the worst consumption sequence. And for $n = 1, 2, 3, \ldots$,

$$W_{n+1}(k_0) = \max_{\{c \geq 0, \, k \geq 0\}} V(c, W_n(k)), \quad \text{where } c + k = g(k_0), \quad 0 \leq k_0 \leq \bar{k}. \tag{A-2}$$

Clearly, (A-1) can be rewritten as

$$W_1(k_0) = V(g(k_0), \underline{U}), \tag{A-2}$$

which is continuous in k_0 and is bounded. It can also be shown that $W_n(k_0)$ is continuous and bounded.

The sequence $\{W_n(k)\}$ defined above can be looked at as a successive approximation of the maximum welfare function $W(k)$. It should be emphasized that $\{W_n(k)\}$ which depends only on $g(k)$ and $V(c, U)$ can be iteratively calculated by using, say, digital computer. Therefore, if we can prove that this sequence converges to the desired function $W(k)$, we can assert that our optimal growth model becomes self-contained in the sense that any optimal program can be determined only from the given functions $g(k)$ and $V(c, U)$.

[18] See, for example, Bellman [2].

Inductively substituting $W_m(k)$, $1 \leq m < n$, into the definition of $W_n(k)$, we get

$$\begin{aligned} W_n(k_0) &= \max_{\{c_1, k_1\}} V(c_1, \max_{\{c_2, k_2\}} V(c_2, \ldots, \max_{\{c_n, k_n\}} V(c_n, \underline{U}), \ldots)) \\ &\qquad \text{where } c_t + k_t = g(k_{t-1}), \quad c_t \geq 0, \quad k_t \geq 0, \quad (t = 1, 2, \ldots, n) \\ &= \max_{\{c_t \geq 0, \, k_t \geq 0\}} U({}_1C_n), \quad \text{where } c_t + k_t = g(k_{t-1}) \quad (t = 1, 2, \ldots, n). \end{aligned} \tag{A-3}$$

Here ${}_1C_n$ represents a terminable consumption sequence $(c_1, c_2, \ldots, c_n, 0, 0, \ldots)$. In words, $W_n(k_0)$ represents the maximum utility level attainable from a given capital stock k_0 when the consumption levels after n periods are constrained to be zero. Therefore, it is easy to show that $W_n(k_0)$ is monotonically nondecreasing with respect to n for any $0 \leq k_0 \leq \bar{k}$. Moreover, $W_n(k_0)$ is uniformly bounded above by the true maximum welfare function $W(k_0)$ that is the maximum of $U({}_1C)$ without any additional constraints on the future consumption sequences other than the technological constraints. So the sequence of bounded and continuous functions $\{W_n(k_0)\}$ converges uniformly to a continuous function $W_\infty(k_0)$ for any $0 \leq k_0 \leq \bar{k}$. That is,

$$W(k_0) \geq W_\infty(k_0) = \lim_{n \to \infty} W_n(k_0) > -\infty. \tag{A-4}$$

Now we want to show that $W(k_0)$ is, in fact, equal to $W_\infty(k_0)$. To see this, let us consider an arbitrary consumption sequence ${}_1C$ feasible for a given initial capital stock $0 \leq k_0 \leq \bar{k}$. For any $n > 0$, a terminable consumption sequence ${}_1C_n'$, of which the first n components are the same as ${}_1C$ and the remaining ones are constrained to be zero, is feasible for k_0 simply by disposing of all the capital stock at the end of period n. Hence, by (A-3), we have for all n

$$W_n(k_0) = \max U({}_1C_n') \geq U({}_1C_n'). \tag{A-5}$$

Since the set of feasible consumption sequences is bounded, closed and convex, ${}_1C_n'$ converge to ${}_1C$ as n goes to infinity. Therefore, by continuity of the stationary ordinal utility function, $U({}_1C_n')$ also converges to $U({}_1C)$, leading to the following inequality.

$$W_\infty(k_0) \geq U({}_1C) \tag{A-6}$$

for all feasible ${}_1C$, including the optimal consumption sequence. Hence, we get

$$W_\infty(k_0) \geq W(k_0) = \max U({}_1C). \tag{A-7}$$

If we combine (A-4) and (A-7), we can assert that the successive approximation: $\{W_n(k_0)\}$ uniformly converges to the true maximum welfare function $W(k_0)$.

$$\lim_{n\to\infty} W_n(k_0) = W_\infty(k_0) = W(k_0) \qquad \text{for} \quad 0 \leqslant k_0 \leqslant \bar{k}. \qquad \text{(A-8)}$$

Note that, since $W_n(k)$ is continuous, (A-8) also proves continuity of $W(k)$ with respect to k. Thus, by successively computing $W_n(k_0)$, we can in principle determine the functional form of $W(k_0)$ to any desired degree of accuracy. But how rapidly this successive approximation converges to $W(k_0)$ is still an open question.

Finally, once we got the function $W(k)$, we can immediately determine the functional form of the reduced utility function from its definition: $\Phi(c, k) = V(c, W(k))$. Since both $V(c, U)$ and $W(k)$ are continuous functions, $\Phi(c, k)$ is also continuous with respect to c and k as was claimed in the text.

Acknowledgment

I am grateful to Professor Kazuo Sato and Professor Paul Samuelson for helpful suggestions and to Professor Peter Diamond, Professor Franklin Fisher, Professor Robert Solow and Mr. Tsuneo Ishikawa for their comments. I am also indebted to the referee of this journal for his detailed comments, suggestions and criticisms. However, the remaining errors are exclusively mine.

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Printed by the St Catherine Press Ltd., Tempelhof 37, Bruges, Belgium.