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Fiat Money and Aggregate Demand Management in a Search Model of Decentralized Exchange

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Abstract

The present paper is an attempt to answer one of the most old-fashioned problems in monetary economics -- why individuals accept a useless piece of paper, called fiat money, in exchange for useful goods? It develops a search-theoretic model of decentralized exchange process and demonstrates that fiat money is accepted by everybody merely because it is believed to be accepted by everybody else in the economy. The paper also studies the macroeconomic implications of this bootstrap explanation of fiat money. It not only reproduces the traditional Keynesian argument for active aggregate demand management when nominal prices are sticky but also suggests a non-Keynesian justification of aggregate demand management whose effectiveness is independent of the degree of price flexibility.

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1. Fiat money and macroeconomics

> The great thinkers of antiquity, and following them a long series of the most eminent scholars of later times up to the present day, have been more concerned than with any other problem of our science with the explanation of the strange fact that a number of goods (gold and silver in the form of coin, as civilization develops) are readily accepted by everyone in exchange for all other commodities, even by persons who have no direct requirements for them or whose requirements have already been fully met.[1]

It was more than a hundred years ago that Carl Menger wrote the above sentence in his Principles of Economics. Whether it is the outcome of further development of civilization or not, gold and silver have since been superseded by a useless piece of paper, called "fiat money", which is issued by government without any promise of convertibility into a commodity. Yet, economists are still searching for a theoretical framework which is capable of explaining the same strange fact that such a mere piece of paper is readily accepted by everyone in exchange for all other useful commodities.

In contemporary economies it is this fiat money that serves as the most important instrument for macroeconomic management. Fiscal policy is exercised through changes in its flow magnitude and monetary policy through changes in its stock magnitude. With a pressing need to understand the effectiveness of macroeconomic policies but without a fully articulated model to explain the circulation of fiat money as a medium of exchange, economists have in recent years had recourse to two short-cut theoretical devices. One of them, due to Clower, is to superimpose on individual demand behavior a "cash-in-advance constraint" -- a constraint which forces every individual to use fiat money as a medium of exchange.[2] Another is Samuelson's consumption-loan model, whose "overlapping generation structure" allows fiat money to have a positive value as a store of wealth, though not as a medium of exchange.[3] In fact, both approaches have proved quite successful in bringing out in full relief the fundamental difference between decentralized monetary economies and the textbook Walrasian theory, and have now become two of the most useful modeling devices for macroeconomic analysis. And yet, it must be recognized that either of them is incomplete as a theory of the medium of exchange function of fiat money, and their success has further enhanced the need for such theory. Macroeconomics is still in search of its theoretical foundation.

The present paper is an attempt to provide some elements for such foundation. Our first objective is to develop a search-theoretic model of decentralized exchange, which is capable of explaining the mechanism through which an intrinsically useless and absolutely inconvertible piece of paper emerges as the universal medium of exchange. Indeed, we are able to show within a formal search model that fiat money is accepted by everybody as a medium of exchange merely because it is believed to be accepted by everybody else as a medium of exchange. Such "bootstrap mechanism" allows fiat money to have a positive value in one of possible exchange equilibria, independently of the "real" conditions of technology and preferences. Moreover, once fiat money gains currency as the universal medium of exchange, it in turn forces everybody to acquire it in order to obtain the good in need. A cash-in-advance constraint thus arises naturally, not as a superimposed constraint, but as an endogenous feature of one of exchange equilibria of the economy.

Our second objective in the present paper is to explore the macroeconomic implications of this search-theoretic explanation of the phenomenon of fiat money. Employing some simple examples, we are able to reproduce the standard macroeconomic results. When by some centralized mechanism nominal prices are determined endogenously, the economy automatically balances its aggregate demand and supply. When nominal prices are determined exogenously (and there being a theoretical presumption for their built-in stickiness in our model), the economy is incapable of balancing aggregate demand and supply and requires some kind of "Keynesian" aggregate demand management for its better performance. But the fact that fiat money is nothing but the product of a bootstrap mechanism, or of self-fulfilling beliefs, leads us to state another case for active government policies. We conclude the present paper by suggesting the possibility of "Non-Keynesian" aggregate demand management, which works through its impacts on the way people form their beliefs and the effectiveness of which is independent of the stickiness of nominal prices.

2. The basic search model of decentralized exchanges

Since the model of decentralized exchange to be worked out in the present paper is taken from our preceding paper [1988], we begin by reviewing its basic framework.[4] We consider an economy with a large number of differentiated goods and a large number of heterogeneous individuals. An individual enters the economy with a given endowment of one unit of one good but with a given need of consuming another good. It is therefore necessary for each individual to search for exchange opportunities in order to obtain the good to consume. Let there be N+1 goods indexed by 0, 1, ...,N.[5] We then suppose that the economy has a decentralized exchange structure which consists of N(N+1)/2 separate trading zones.[6] Each trading zone is specialized to the exchanges between only one pair of goods and is distinguished by the indices (i,j) of the goods to be exchanged. (Hence, (i,j) zone is identical with (j,i) zone.)

An individual who currently holds good i but needs to consume good j may then decide to visit (i,j) trading zone to search for a mirror-symmetric individual who is willing to exchange good j for good i. As soon as such a trading partner is found, he will exchange a unit of good i for a unit of good j and retire from the economy for consumption. (We assume until section 8 that the terms of trade between any pair of goods is fixed at one to one.) This is the strategy of direct barter, the success of which requires double coincidence of wants. The same individual may, however, choose a roundabout route. He first visits (i,k) trading zone (with $k \neq j$) and searches for an individual willing to exchange good k for good i. Having found such a partner and obtained a unit of good k, he then makes a second trip to (k,j) trading zone and searches for an individual willing to exchange good j for good k. He will retire from the economy only after he succeeds in this second exchange and obtains a unit of good j. This is the strategy of indirect exchange which uses good k as a medium of exchange. The same individual may even seek a longer sequence of indirect exchanges which uses two or more media of exchange.

We specify the meeting process in each trading zone in the following manner. Denote by q_{ji} the frequency (relative to the total population) of individuals who are willing to supply good j in exchange for good i in (i,j) trading zone, and call it "supply-demand frequency". (We have $q_{ii} = 0$ and $\Sigma_j \Sigma_i q_{ji} \leq 1$.[7]) We then assume that the probability for each individual of finding one of j-supplying, i-demanders in the (i,j) trading zone is equal to $q_{ji}dt$ in a small time interval dt.[8] For each individual the meeting process thus constitutes a Poisson process with the meeting rate equal to the frequency of trading partners. This appears to be a reasonable assumption about the way people meet each other in a sparsely populated "zone", though it is certainly a questionable description for a congested zone or an organized trading "post".[9] We shall also suppose throughout the present paper that every individual in the economy is in his expectational equilibrium in the sense that his subjective estimates of these supply-demand frequencies are identical with their objective values.

To complete the description of the individual search process, denote by u the utility gain from consuming the needful good, by b the cost or the utility loss from engaging in exchange activity, and by c the cost or utility loss from spending one time unit in search activity. Though not essential to the main results of the present paper, these parameters are taken to be constant over time and uniform across individuals and goods. We then assume that each individual in this economy chooses his search strategy so as to maximize the undiscounted expected value of life-time utility.[10]

Indeed, if we denote by V_{ij} the maximum value of the expected life-time utility of an i-holding, j-consumer and if we use an obvious convention that $V_{jj} = u$, a simple dynamic programming argument would enable us to deduce the following functional equation, as long as all the demand-supply frequencies $\{q_{ji}\}$ remain constant over time:

$$(1) \quad V_{ij} = \max_{k}[\, V_{kj} - b - c/q_{ki} \,] .$$

Here, V_{kj} is the maximum utility that an j-consuming individual is expected to obtain when he holds good k instead of good i, b is the exchange cost required for obtaining good k in exchange of good i, and c/q_{ki} is the expected search cost for meeting one of k-supplying, i-demanders in (i,k) trading zone. (The third term c/q_{ki} is nothing but c the search cost per unit time multiplied by the expected search time $1/q_{ki}$.) Hence, the expression $V_{kj}-b-c/q_{ki}$ in the right-hand-side bracket represents the expected net utility of visiting (i,k) trading zone, and our fellow individual chooses an index k which promises him the highest net utility for his life-time.[11] In fact, if the optimum k turns out to be j, our fellow individual will seek a direct barter in (i,j) zone (with $V_{jj} = u$); if the optimum k becomes different from j, he will seek an indirect exchange using good k as a medium of exchange in (i,k) zone.

Finally, we suppose for the sake of simplicity that the utility of autarky (or of not consuming one's necessity) is $-\infty$ and that our fellow individual takes the trouble to search for trading opportunities as long as a finite utility is expected from search activity.[12] Hence, we have

(2) $V_{ij} > -\infty$,

as the condition for staying in the economy.

3. On the notion of fiat money equilibrium

We now turn to the description of possible equilibrium patterns of exchange process in our economy. For this purpose we have to introduce two representations of the economy's demographic structure, lying at the level successively deeper than the set of supply-demand frequencies $\{q_{ij}\}$ we introduced in the preceding section. First, denote by f_{ij} the relative frequency of individuals who currently hold good i but needs to consume good j, and call it "supply-need frequency". (We have $f_{ii} = 0$ and $\Sigma_i \Sigma_j f_{ij} = 1$.) Second, denote by e_{ij} the relative frequency of individuals who were born with an endowment of good i but with a need to consume good j, and call it "endowment-need frequency". (We have $e_{ii} = 0$ and $\Sigma_i \Sigma_j e_{ij} = 1$.) We shall suppose in what follows that, as soon as one of the i-endowed, j-consumers retires from the economy, a new individual with the same endowment-need constitution enters the economy. This simple parent-child succession mechanism will keep the set of endowment-need frequencies $\{e_{ij}\}$ constant over time, independently of the on-going exchange process within the economy.

We have thus three sets of frequencies at hand, each corresponding to a different layer of the economy's demographic structure. The surface layer consists in the set of supply-demand frequencies $\{q_{ij}\}$, which is directly observable to every individual searching in it. The second layer consists in the set of supply-need frequencies $\{f_{ij}\}$. It describes the distribution among individuals of the behavioral characteristics, which determine the "form" of their search strategies. And the last and the deepest layer consists in the set of endowment-need frequencies $\{e_{ij}\}$. It summarizes the invariant distribution of "technology and preferences" or the economy's "fundamentals". Among these three frequencies, the relationship between $\{q_{ij}\}$ and $\{f_{ij}\}$ is relatively simple to examine. We only have to count the former in terms of the latter. But when we come to the analysis of the relationship between $\{f_{ij}\}$ and $\{e_{ij}\}$, the mere counting of the existing frequencies no longer suffices. We need to study a chance mechanism which governs the meeting processes among different individuals who are searching for the opportunities to exchange, and then trace out their dynamic impacts upon the evolutionary history of $\{f_{ij}\}$. We therefore postpone the analysis of the relationship between $\{f_{ij}\}$ and $\{e_{ij}\}$ to section 5, and concentrate on the analysis of the relationship between $\{q_{ij}\}$ and $\{f_{ij}\}$ until then.

Let us begin by "counting" each of $\{q_{ij}\}$ in terms of $\{f_{ij}\}$. When i-supplying, j-consumers have decided to barter good i for good j, they directly visit (i,j) trading zone and add their frequency f_{ij} to q_{ij}. And when i-supplying, k-consumers (with $k \neq j$) have decided to use good j as a medium of exchange, they also visit the (i,j) trading zone and add their frequency f_{ik} to q_{ij}. Since the condition for i-supplying, j-consumers to choose to barter is given by $V_{ij} = V_{jj}-b-c/q_{ji}$ ($= u-b-c/q_{ji}$) $> -\infty$ and the condition for i-supplying, k-consumers to demand good j as a medium of exchange is given by $V_{ik} = V_{jk}-b-c/q_{ji} > -\infty$, we can easily sum up these calculations as follows.

$$(3) \quad q_{ij} = \sum_{\{k:\ f_{ik} > 0 \text{ and } V_{ik} = V_{jk}-b-c/q_{ji} > -\infty\}} f_{ik} .$$

We have come full circle. Given a set of supply-demand frequencies every individual in the economy determines his search strategy in order to maximize his expected life-time utility (1). But, as we have just seen in (3), the set of supply-demand frequencies in itself is no more than the aggregate outcome of every individual's search activity. Our exchange economy is able to find itself in equilibrium only if these micro and macro relations turn out to be all consistent with each other. Formally, we have

<u><Definition>: An economy is said to be in a state of "exchange equilibrium" (relative to a given set of supply-need frequencies $\{f_{ij}\}$) if the set of supply-demand frequencies $\{q_{ij}\}$ satisfies both (1) and (3).</u>

In fact, our economy is able to accommodate, at least potentially, a large number of different forms of exchange equilibrium. In our preceding paper [1988] we introduced two of the most familiar forms of exchange equilibrium -- "barter equilibrium" and "monetary equilibrium".[13] Barter equilibrium was defined to be a form of exchange equilibrium in which every active individual seeks a barter exchange; monetary equilibrium a form of exchange equilibrium in which every active individual uses a particular good as the exclusive medium of exchange. One of our central theses was that, while barter equilibrium does not always exist, any well-defined economy has as many monetary equilibria as the number of goods. Barter exchange requires a double coincidence of wants -- a situation in which one individual has the good that the other individual needs and needs the good that the other individual has. Monetary exchange is, in contrast, a "social" process which is able to support itself without any conditions on the distribution of "technology and preferences".

The present paper continues the same line of investigation but focuses on the purest form of monetary equilibrium which circulates as "fiat money" a piece of intrinsically useless paper issued by the state. We now introduce

<Definition>: An exchange equilibrium is said to be a "monetary equilibrium with fiat money" or a "fiat money equilibrium" for short if every active individual uses as the exclusive medium of exchange an intrinsically useless piece of paper issued by the state without any convertibility into a commodity and without any legal enforcement of its circulation.[14]

There is indeed little mystery in the use of a useless paper as fiat money, as long as the state guarantees its full convertibility or enforces its public circulation.[15] The real mystery, if any, is why and how does such mere paper acquire the power to circulate among individuals as money even without any state guarantee and without any state legislation.

4. Fiat money equilibrium in the short-run

Let us designate by index 0 the useless paper issued by the state in order to distinguish it from the set of all the other "real" goods {1, 2, ...N} which serve as the objects of consumption for some private individuals in the economy. We have $e_{i0} = 0$ and $e_{0i} = 0$ for any i for this "fictive" part of the economy. We now have to introduce an assumption about the way endowment-need frequencies are distributed in the "real" part of the economy.

<u><Assumption 1>: An economy is "connected" in the sense that for any two indices of real goods i ($\neq 0$) and j ($\neq$ 0 and i) there always exists a connecting sequence of strictly positive endowment-need frequencies such that $e_{ik} > 0$, $e_{kl} > 0$, ..., $e_{gh} > 0$ and $e_{hi} > 0$.</u>

This is a minimum requirement for an economy to be meaningfully called an economy. If there is an almighty goddess in (or above) a connected economy, she is able to feed every individual by suitable series of exchanges of endowments without importing anything from outside. To us, mortal beings, however, the problem is whether there is any decentralized exchange process which is able to achieve the same end without any transcendental intervention.

The first question we have to ask is: Under what conditions does every individual come to "voluntarily" use a piece of paper as money even if it has no "use" value in itself? The following proposition provides us with a complete characterization of such conditions.

<u><Proposition 1>: Every possible individual in the economy voluntarily uses a paper of index 0 as the exclusive medium of exchange if and only if the set of supply-demand frequencies $\{q_{ij}\}$ satisfies the following set of inequalities.</u>

(4a) $2b + c/q_{j0} + c/q_{0i} < \infty$ for any i ($\neq$ 0) and j ($\neq$ 0 and i) ,

(4b) $2b + c/q_{j0} + c/q_{0i} < b + c/q_{ji}$ for any i ($\neq$ 0) and j ($\neq$ 0 and i) ,

(4c) $2b + c/q_{j0} + c/q_{0i} < 2b + c/q_{jk} + c/q_{ki}$ for any i ($\neq$ 0), j ($\neq$ 0 and i) and k ($\neq$ 0, i and j) ,

(4d) $b + c/q_{j0} \leq 2b + c/q_{jk} + c/q_{k0}$ for any j ($\neq$ 0) and k ($\neq$ 0 and j) .

Since the proof of the necessity and sufficiency of these inequalities is a mere adaptation of the one given in our preceding paper [1988], we do not write it down here. Indeed, their economic interpretation is so straightforward that we hardly need any formal proof to convince ourselves of their validity. The first inequality (4a) says merely that it is better to use the state-issued paper as a medium of exchange than to suffer the misery of autarky. The second (4b) says that it is less costly to use the state-issued paper as a medium than to barter directly. The third (4c) says that among all the possible candidates for medium of exchange the state-issued paper is the least costly to use and no other good can rival it. The fourth (4d) then says that once the state-issued paper has come into one's possession it is less costly to exchange it directly for the needful good than to seek another indirect exchange. If all these inequalities are combined together, they certainly induce (or more precisely, "force") every individual to use the state-issued paper as the exclusive medium of exchange or as fiat money.

Having thus characterized the conditions for the universal use of fiat money in terms of the supply-demand frequencies $\{q_{ij}\}$, our next task is to reduce them to the more generic supply-need frequencies $\{f_{ij}\}$.

Suppose now that by some reason everybody in the economy has decided to use the state-issued paper as fiat money. It then follows that only those individuals who happen to hold such paper are able to demand the needful good in its return. A "cash-in-advance constraint" *à la* Clower has thus emerged within the economy and begins to restrict everybody's demand behavior. Hence, the frequency q_{0i} of 0-supplying, i-demanders consists only of the frequency f_{0i} of the active 0-supplying, i-consumers. Since q_{0i} is zero for i with $f_{0i} = 0$, we have

(5a) $q_{0i} = f_{0i}$ for any i ($\neq 0$) .

It also follows that any individual who still holds his endowed good must first seek to exchange it for the state-issued paper in order to obtain his needful good in the second exchange. Hence, all the frequencies f_{ik}'s of active i-supplying individuals now add up to form the frequency q_{i0} of i-supplying, 0-demanders. A "social" demand for the state-issued paper has thus emerged endogenously in the economy, even if there was no "real" demand for it in the beginning. Since the value of q_{i0} is unaffected by the addition of zero frequencies, we indeed have

(5b) $q_{i0} = \sum_k f_{ik}$ for any i ($\neq 0$) .

Finally, in this situation of universal monetary exchange no individual seeks to exchange his endowed good directly for the needful good. The frequency of individuals who simultaneously seek to supply and demand real goods thus disappear from the economy, and we have

(5c) $q_{ij} = 0$ for i ($\neq 0$) and j ($\neq 0$) .

Note that the 0-supplying, i-demanders whose frequency is given by q_{0i} are what we usually call "buyers" of good i and that the i-supplying, 0-demanders whose frequency is given by q_{i0} are what we usually call "sellers" of good i. Indeed, the (i,0) trading zone in which these buyers and sellers of good i search for each other and exchange with each other is precisely what we usually call the "market" for good i.

The question arises: Are these supply-demand frequencies consistent with each other and actually constituting a state of monetary equilibrium? The following Proposition provides us with a first answer to this question.

<Proposition 2>: Suppose $f_{0i} > 0$ and $\sum_{k \neq 0} f_{ik} > 0$ for all i ($\neq 0$). Then,

__the economy has a fiat money equilibrium.__

(Proof): All we have to do is to verify that $\{q_{ij}\}$, given by (5a)--(5c), satisfy the set of inequalities (4a)--(4d) stipulated in Proposition 1. Since the proof is again a simple adaptation of that of Proposition 6 of our preceding paper [1988], we only exhibit here the inequality (4b) as an illustration. This is indeed a trivial thing to do. First, (5c) implies that for any i ($\neq 0$) and j($\neq 0$) there is no individual in (i,j) trading zone. Hence, the expected cost of barter, $b+c/q_{ji}$, becomes equal to ∞. Second, the supposed conditions, along with (5a) and (5b), imply that the probability of meeting a seller in (i,0) trading zone and the probability of meeting a buyer in (j,0) trading zone are both strictly positive. Hence, the expected cost of using the state-issued paper as money, $2b+c/q_{j0}+c/q_{0i} = 2b+c/\{\Sigma_{k\neq 0} f_{jk}\}+c/f_{0i}$, becomes finite. We therefore obtain the inequality (4b) for any i ($\neq$ 0) and j($\neq$ 0 and i), which says that it is less costly to use the state-issued paper as money than to barter directly. (QED)

What we have seen in the above Proposition is nothing but the working of a "bootstrap" mechanism. Everybody in the economy abandons a barter trade and uses the state-issued paper as money, merely because everybody else abandons a barter trade and uses the state-issued paper as money. The state does not have to force the circulation or even guarantee the convertibility. A mere paper is able to lift itself to the status of money and have a positive value by its own bootstraps!

The above Proposition 2, however, has failed to remove all the "real" constraints from fiat money equilibrium. Its existence still requires the support of two conditions: $q_{0i} = f_{0i} > 0$ and $q_{i0} = \Sigma_{k\neq 0} f_{ik} > 0$ for all i ($\neq$ 0), of which the first one says that for any good i ($\neq$ 0) there must be at least one individual (that is, one buyer) who is willing to offer a piece of state-issued paper in its exchange and the second one says that for any good i ($\neq 0$) there must be at least one individual (that is, one seller) who is willing to demand a piece of state-issued paper in its exchange. Unless these two conditions themselves are shown to emerge naturally within the economy, our claim for the bootstrap nature of fiat money equilibrium remains far from complete.

This in view, let us turn to the study of how the supply-need frequencies $\{f_{ij}\}$ are determined in the economy when the state issued paper is used by everybody as fiat money. Indeed, we now have to examine the circulation of money and individuals from the dynamical standpoint.

5. Dynamics of monetary exchange process

Suppose again that by some reason every individual in an economy has decided to use a piece of state-issued paper as fiat money. (We do not know at this point whether it is a voluntary act or not.) Every piece of fiat money is then handed over from one individual to another as the medium of exchange and circulates continuously within the economy. Let us denote by M the "per capita" stock of fiat money and suppose that its level is fixed by the state once and for all at some initial period.[16] (Because of the assumed unity of all the nominal prices, this M represents the "real" per capita quantity of money.) Since each individual holds money solely for the purpose of transaction and carries only one piece of it at a point in time, the level of M determines the total frequency of money-holding individuals who are able to buy the goods they need for consumption. We thus have

(6a) $\sum_{i \neq 0} q_{0i} = \sum_{i \neq 0} f_{0i} = M$.

Furthermore, if we note the adding-up condition for frequencies, we can also calculate the total frequency of individuals who are currently holding non-monetary goods in order to sell them later.

$$\text{(6b)} \quad \sum_{i\neq 0} q_{i0} = \sum_{i\neq 0} \sum_{k\neq 0} f_{ik} = 1 - M .$$

What the per capita stock M of fiat money determines is, therefore, not the total frequency of buyers *per se* but the relative balance between the aggregates of buyers and sellers.

In order to exclude trivial situations, we need to introduce:

<u><Assumption 2>: 0 < M < 1.</u>[17]

In other words, the amount of fiat money issued by the state is neither too small nor too large, so that there are at least one individual who is currently holding fiat money and another who is currently not holding fiat money.

Let us now shift our attention from aggregative to disaggregative level and study the "birth and death" process of the frequency f_{ij} of the sellers of good i ($\neq$ 0) whose ultimate objective is to consume good j ($\neq$ 0 and i). On the one hand, it is easy to see that one of these i-selling, j-consumers disappears from (i,0) trading zone whenever he meets a buyer of good i and transforms himself into a buyer of good j. Since his probability of meeting a buyer of good i is equal to $q_{0i}dt$ in a small time interval dt, the total "death" probability of all the i-selling, j-consumers can be calculated as $f_{ij}q_{0i}dt$. On the other hand, a new i-selling, j-consumer steps into (i,0) trading zone as a successor whenever one of the i-endowed, j-consumers who hold fiat money actually meets a seller of good j in (j,0) trading zone and retires from the economy for consumption. Since the frequency of these i-endowed, j-buyers is equal to $e_{ij}-f_{ij}$ and the probability for each of them to meet a seller of good j is $q_{j0}dt$ in a time interval dt, we can calculate the total "birth" probability of i-selling, j-consumers as $(e_{ij}-f_{ij})q_{j0}dt$. The law of large numbers then allows us to write down the following dynamical equation, which accounts for the "net" increase or decrease in f_{ij} for each i ($\neq$ 0) and j ($\neq$ 0 and i).

$$\dot{f}_{ij} = -q_{0i}f_{ij} + q_{j0}(e_{ij} - f_{ij}) \quad \text{for any } i\ (\neq 0) \text{ and } j\ (\neq 0 \text{ and } i).$$

where " $\cdot$ " denotes the time derivative d/dt. Substituting (5a) and (5b), we can rewrite the above system of dynamical equations all in terms of $\{f_{ij}\}$.

$$(7) \quad \dot{f}_{ij} = -\{\sum_{k\neq 0}(e_{ki} - f_{ki})\}f_{ij} + (\sum_{k\neq 0} f_{jk})(e_{ij} - f_{ij}) \quad \text{for } i\ (\neq 0) \text{ and } j\ (\neq 0 \text{ and } i).$$ [18]

It should be noted here that once the evolution of the supply-need frequencies of sellers, $\{f_{ij}\}$ for i ($\neq$ 0) and j ($\neq$ 0 and i), is stipulated by the above dynamical system, that of the supply-need frequencies of buyers, $\{f_{0i}\}$ for i ($\neq$ 0), can be traced as mere residuals. It is because each individual not selling his endowed good must be buying a needful good and we have the following relationship between the frequencies of sellers and buyers.

$$(8) \quad f_{0i} = \sum_{k\neq 0}(e_{ki} - f_{ki}) \quad \text{for any } i\ (\neq 0).$$

We have now deduced all the necessary equations for the study of the dynamical evolution of supply-need frequencies when the economy has by some reason decided to use the state-issued paper as fiat money. In fact, in view of (8), we only have to examine the frequencies of sellers, $\{f_{ij}\}$ with i ($\neq$ 0) and j ($\neq$ 0 and i), whose dynamical motions are described by the system of (N-1)N differential equations (7), together with an adding-up equation (6b).

What is remarkable about this dynamical system is that in spite of its apparent complexity it is quite well-behaved. Let us denote by $\{f^*_{ij}\}$ a set of "steady-state" frequencies of sellers, which can be determined by setting each of the right-hand-side of (7) equal to zero. Rearranging terms, we have

$$(9)\quad f^*_{ij} = \frac{\Sigma_k f^*_{jk}}{\Sigma_k f^*_{jk} + \Sigma_k (e_{ki} - f^*_{ki})} e_{ij} \quad \text{for } i\ (\neq 0) \text{ and } j\ (\neq 0 \text{ and } i)\ .$$

Though it is in general not possible to obtain a closed-form expression for the steady-state frequencies $\{f^*_{ij}\}$, the following proposition tells us that the dynamical system of $\{f_{ij}\}$ has almost all the desirable properties that a dynamical system can have.

<Proposition 3>: Suppose that every individual in a connected economy is using the state-issued paper as fiat money with $0 < M < 1$. Then, the set of supply-need frequencies of sellers, $\{f_{ij}\}$ for i ($\neq 0$) and j ($\neq 0$ and i), will in the long-run converge to a unique steady-state $\{f^*_{ij}\}$ determined by (9). Moreover, for any i and j such that $e_{ij} > 0$ the value of f_{ij} will never hit the boundaries, 0 and e_{ij}, at least after some finite period of time.

Since the proof is quite involved, it is relegated to Appendix 1 below.

Once we have succeeded in characterizing the well-behaved dynamical performance of the set of supply-need frequencies $\{f_{ij}\}$ in this manner, it becomes a matter of simple translation to characterize the dynamical performance of the set of supply-demand frequencies $\{q_{ij}\}$. Denote by q^*_{0i} the steady-state frequency of the buyers of good i and by q^*_{0i} the steady-state frequency of the sellers of good i. (Because of (5c), all the other supply-demand frequencies are zero.) Taking note of the relation (8) and substituting (9) into (5a) and (5b), we can express their values implicitly as follows.

$$(10a)\quad q^*_{i0} = \sum_{k\neq 0} f^*_{ik} = \sum_{k\neq 0} \frac{q^*_{k0}}{q^*_{k0}+q^*_{0i}} e_{ik} \qquad \text{for } i\ (\neq 0)\ ,$$

$$(10b)\quad q^*_{0i} = f^*_{0i} = \sum_{k\neq 0}(e_{ki} - f^*_{ki}) = \sum_{k\neq 0} \frac{q^*_{0k}}{q^*_{i0}+q^*_{0k}} e_{ki} \qquad \text{for } i\ (\neq 0)\ .$$

We then have a complete description of the dynamics of $\{q_{0i}\}$ and $\{q_{0i}\}$.

<u><Proposition 4>: Suppose that every individual is using the state-issued paper as fiat money in a connected economy with $0 < M < 1$. Then, both the buyers' frequencies $\{q_{0i}\}$ and the sellers' frequencies $\{q_{i0}\}$ will assume positive values at least after some finite period of time and will in the long-run approach their unique steady-state values $\{q^*_{0i}\}$ and $\{q^*_{i0}\}$, respectively determined by (10a) and (10b).</u>

The Proof of this proposition is relegated to the last part of Appendix 2.

6. Pure bootstrap nature of fiat money equilibrium

Indeed, we have not only succeeded in giving a complete description of the dynamical evolution of the frequencies of individuals when by some reason they all come to use the state-issued paper as fiat money, but also succeeded in demonstrating the very existence of a fiat money equilibrium just as a free by-product. At the end of section 4, we noted that our Proposition 2 just fell short of establishing the complete bootstrap nature of fiat money equilibrium. Because its existence still required two conditions, $q_{0i} = f_{0i} > 0$ and $q_{i0} = \Sigma_{k \neq 0} f_{ik} > 0$ for all i $(\neq 0)$, which say that, unless there are at least one buyer and one seller for each good in the economy, the use of fiat money cannot be sustained as a voluntary outcome of decentralized exchange processes. But what we have now seen in Proposition 4 is that the very process of monetary exchange gives natural birth to such individuals! In fact, the emergence of a seller of each good is immediate because the universal adoption of the state-issued paper as fiat money forces every individual to seek it as the medium of exchange and turn himself into a seller of his endowed good. (This is what the counting equation (5b) means.) The connectedness of the economy then guarantees that for each good there is at least one individual who was born with it. The emergence of a buyer of each good, on the other hand, may take some time. But, as long as there is one piece of fiat money in the economy, its circulation among individuals through the repeated sequence of purchase and sale will eventually bring to any individual a buyer of his endowed good. He will then be able to obtain fiat money in return and transform himself into a buyer of the good he needs to consume. The connectedness of the economy again guarantees that for each good there is at least one individual who was born with a real need for it. We thus have

<Proposition 5>: As long as the economy is connected and $0 < M < 1$, fiat money equilibrium is always able to support itself by its own bootstraps.

We have finally succeeded in demonstrating the bootstrap nature of fiat money without any qualifications. A piece of state-issued paper is both demanded and supplied as fiat money merely because it is used as fiat money by every individual in the economy. It is the purest form of "social contrivance" which is neither the creature of the "technology and preferences" nor the creature of the "state". Having thus established a search-theoretic foundation of monetary economics, we are now in a position to make some attempt to construct a framework for macro-economic analysis.

7. "Keynesian" non-neutrality of money in a fixed price economy

The fact that fiat money can circulate in the economy without any legal enforcement does not necessarily mean that its actual function is beyond the control of the state. On the contrary, it is usually the state which has the monopoly power to adjust the supply of fiat money in order to better the performance of macroeconomy.

Denote by Y^d the per capita value of aggregate demand. Since all the terms of trade have been assumed to be unity, this can be calculated by summing all the frequencies of buyers $\{q_{0i}\}$ in the economy. But by (6a) this is also equal to the real per capita stock of fiat money M. We thus have

$$(11a) \quad Y^d = \sum_{i \neq 0} q_{0j} = M .$$

Next, denote by Y^s the per capita value of aggregate supply. Its magnitude can be calculated by summing all the frequencies of sellers $\{q_{i0}\}$, but by (6b) this turns out to be equal to 1-M. We thus have

$$(11b) \quad Y^s = \sum_{i \neq 0} q_{i0} = 1-M .$$

It is therefore the real per capita stock of fiat money M that determines the relative balance between aggregate demand Y^d and aggregate supply Y^s. The control of real money supply is tantamount to the aggregate demand management in our simple exchange economy. Indeed, it is trivial to see that

$$(12) \quad M \gtreqless 1/2 \quad \Leftrightarrow \quad Y^d \gtreqless Y^s .$$

In words, when the state sets M above (below) 1/2, aggregate demand exceeds (falls short of) aggregate supply, and it is only when M is set equal to 1/2 the economy's aggregate demand and supply are brought into balance with each other. Money is therefore far from neutral in this economy, and there is an ample room for aggregate demand management in it. In view of the assumed fixty of all the prices, this "Keynesian" result is not surprising at all.

But how can we specify the goal of aggregate demand management? A sensible way to do is to set up a certain social welfare function and ask the state to fix real money supply at the level which would maximize its value.[19] In general, however, the heterogeneity of individuals makes it very hard to obtain a tractable result, and we now have to turn to two special examples, whose symmetrical structure will enable us to treat all them uniformly. The first one is an economy with doubly symmetric in endowments and needs, or

(13a) $e_{ij} = e_{ji} = 1/N(N-1)$ for any i ($\neq 0$) and j ($\neq 0$ and i) .

This economy is abundantly connected. The second example is an economy whose distribution of endowments and needs does not allow any double coincidence of wants but still is minimally connected, or

(13b) $e_{12} = e_{23} = \ldots = e_{N-1,N} = e_{N1} = 1/N$; and all the other e_{ij}'s = 0.

In our previous paper [1988] it was shown that while the first economy has a barter equilibrium the second has none. Needless to say, we now know that both have a fiat money equilibrium as does any other connected economy.

Substituting the above examples into (10a) and (10b), we can explicitly calculate the steady-state frequencies of buyers and sellers, q^*_{0i} and q^*_{i0}. Indeed, we obtain the same expressions in both cases.

(14a) $q^*_{0i} = M/N$; (14b) $q^*_{i0} = (1-M)/N$ for all i ($\neq 0$) .

This is the simplest possible demand and supply functions under the "cash-in-advance constraint". If we substitute the above symmetric steady-state solution into the expected life-time utility of a representative individual, we can easily express it as

(15) $V_{ij} = u - 2b - cN/M - cN/(1-M)$.

It is then trivial to see that this expression would be maximized when M is set equal to 1/2. But by (12) this is precisely the level of money supply that brings a balance between aggregate demand and aggregate supply. We thus have the following "Keynesian" proposition.

<Proposition 6>: In the special examples of doubly symmetric endowment-need distribution (13a) and minimally connected endowment-need distribution (13b), the representative individual's expected life-time utility is maximized when the state sets the per capita quantity of fiat money M equal to 1/2 or equal to the level which balances aggregate demand Y^d and aggregate supply Y^s.

In the more general situations, the optimum per capita quantity of money may deviate from the balanced number 1/2, but it still appears to be a good rule of thumb as the target supply level of fiat money.[20]

8. "Classical" neutrality of money in a flexible price economy

Thus far we have supposed that terms of trade between good and fiat money, or which comes to the same thing, every nominal price is fixed at unity in our search economy. Let us make some attempt to remedy this assumption.

The first thing we have to note is that any nominal price, as long as it is uniform across all goods, is consistent with fiat money equilibrium. Let this uniform price be p, which may be greater or smaller than unity. Then, all the indirect exchanges between two goods are going to be mediated not by one unit but by p unit of fiat money. As far as the "real" structure of the economy is concerned, everything is the same as before, except for the fact that the value of money now becomes 1/p and the real quantity of fiat money must now be written explicitly as M/p. Is there any other price structure that is consistent with fiat money equilibrium?

When a buyer and a seller meet each other in a trading zone, their gains from exchange are in general unequal. In (i,0) trading zone, for instance, a buyer of good i is able to gain a utility equal to $(u-b) - (u-b-c/q^*_{i0}) = c/q^*_{i0}$ and a seller of good i is able to gain from the same exchange a utility equal to $(u-2b-c/q^*_{j0}) - (u-2b-q^*_{j0}-c/q^*_{0i}) = c/q^*_{0i}$. When $c/q^*_{i0} > c/q^*_{0i}$, the buyer is more to lose from the failure of exchange and finds himself in a weak bargaining position; when $c/q^*_{i0} < c/q^*_{0i}$, the buyer is less to lose and is in a strong bargaining position. In either situation there appears to be a room for price negotiation in the trading zone.

However, as long as each individual is endowed with only one unit of a pre-determined good at birth and is able to store only one unit at a time, no degree of freedom is left for each trading pair to agree on the terms of their own trade by their own negotiation. To see this, suppose that the same nominal price p is prevailing for all goods except i. Then, every buyer in (i,0) trading zone has p unit of fiat money as the revenue from the first exchange and is able to offer at most that unit to the seller. Every seller in (i,0) trading zone in turn has one unit of endowed good but has to secure at least p unit of fiat money in its return in order to be able to buy one unit of his needful good in the second exchange. Independently of their relative bargaining position, the only agreeable terms of trade between good i and fiat money is p, which prevails over all the other goods in the economy.

Within the (restricted) framework of the present paper, there is therefore a fundamental difficulty in modeling the process of decentralized price negotiations which take place simultaneously in a large number of separate trading zones. Indeed, the above consideration gives us a reasonable presumption for the built-in stickiness of any (uniform) nominal price once it is established by some historical accident or by some conscious decision. And this also gives us a theoretical justification for the "Keynesian" assumption we have maintained so far in this paper.[21]

In spite of all these problems, however, it is also of some interest, perhaps not for its own sake but for the purpose of comparison, to try to impose a game-theoretic model of price-formation on our search model. To this end, we again have to turn to the two special examples (13a) and (13b), whose symmetric structure allows us to do so without violating the built-in uniformity of nominal prices in fiat money equilibrium.

In these symmetric examples it follows from (14a) and (14b) that the steady-state frequencies of buyers and sellers become equal to $q^*_{0i} = M/(pN)$ and $q^*_{i0} = (1-M/p)/N$ when the uniform nominal price is p. We can thus calculate the utility gain from exchange for buyer and seller as $c/q^*_{i0} = c/\{(1-M/p)/N\}$ and $c/q^*_{0i} = c/\{M/(pN)\}$ respectively in each trading zone. If we follow Diamond [1984a] who in turn followed the axiomatic bargaining solution of Nash by assuming that all the buyers and sellers in the economy determine the terms of trade that equalizes their respective utility gains, we indeed obtain the following endogenous price-formation rule.[22]

(16) $c/\{(1-M/P)/N\} = c/\{M/(pN)\}$, or $M/p = 1/2$.

If the nominal price level p is determined (by some centralized manner) in accordance with the Nash bargaining solution, the real per capita supply of fiat money M/p adjusts itself to the value of 1/2. As we have seen in Proposition 6, this is nothing but the optimum number that balances aggregate demand and aggregate supply and at the same time maximizes the representative individual's expected life-time utility. Fiat money now becomes completely neutral, and we have the following "classical" proposition.

<u><Proposition 7>: In the special examples of doubly symmetric endowment-need distribution (13a) and minimally connected endowment-need distribution (13b),</u>

if the uniform nominal price level is determinated by the Nash bargaining solution, the real per capita quantity of fiat money M/p adjusts itself to the level which balances aggregate demand Y^d and aggregate supply Y^s and maximizes the representative individual's expected life-time utility.

9. Plethora of exchange equilibria and the "non-Keynesian" case for aggregate demand management

Having started from the detailed specification of the search-theoretic model of decentralized exchange, we have now ended up with a textbook-like opposition between "Keynesian" non-neutrality and "classical" neutrality of money. When the nominal prices in fiat money equilibrium are exogenously given, the economy is incapable of balancing its aggregate demand and aggregate supply and requires some kind of aggregate demand management to enhance the representative individual's expected life-time utility. When, on the other hand, the nominal prices are endogenously determined in fiat money equilibrium (in accordance with the Nash bargaining solution), aggregate demand and aggregate supply automatically adjust themselves to each other and render the aggregate demand management superfluous. Setting aside the theoretical presumption for the built-in stickiness of nominal prices we pointed out in the preceding section, is the contribution of our search-theoretic model to macroeconomics only to reproduce the well-known propositions at a level as elementary as that of undergraduate textbooks?

The answer to this question is, however, "no", and we use the rest of this section for an elucidation of this hastily voiced negative answer.

The fiat money equilibrium we have so far been concerned with is only one of a large number of possible exchange equilibria that a search economy may settle down. In fact, the same bootstrap mechanism which generates and supports fiat money equilibrium is also capable of generating and supporting as many commodity money equilibria as the number of durable goods in the economy.[23] More important to macroeconomics is the existence of barter equilibrium and autarky equilibrium. When the distribution of endowments and needs is well-balanced, as is the case in the doubly symmetric example of (13a), the economy may find a comfortable niche in a barter equilibrium in which every individual seeks a direct barter without any mediation of money.[24] Moreover, even if the economy is not able to accommodate a barter equilibrium, it always has a autarky as a stable exchange equilibrium. It is an equilibrium situation in which no individual takes any opportunity to enter the economy simply because no other individuals take any opportunities to do so. It is, in other words, an outcome of the reversal working of the bootstrap mechanism. There may also exist a large number of hybrid equilibria which combine fiat money, barter and autarky equilibria in various proportions.

It goes without saying that fiat money equilibrium is Pareto superior to autarky equilibrium. With respect to barter equilibrium, fiat money equilibrium may not have Pareto superiority in general. But, as is shown in Appendix 2, if the number of goods is sufficiently large, it is likely to dominate barter equilibrium.

With the intrinsic multiplicity of exchange equilibria that are often Pareto-rankable, there then arises a case for active government policies. If the fiat money equilibrium has been created by self-fulfilling beliefs or what we have called bootstrap mechanism, it may also be destroyed by another sequence of self-fulfilling beliefs working in the opposite direction. It may at any time collapse into autarky or barter or some other low level equilibrium by an unfortunate string of pessimistic beliefs. Here lies a room for the state intervention. The state is able to prevent the economy from unraveling its fiat money equilibrium or to steer it out of the trap of some low level equilibrium, by directly stimulating aggregate demand through its conscious use of fiscal and monetary policies and thereby restoring the free working of the bootstrap mechanism. This is what we call the "non-Keynesian" aggregate demand management, whose effectiveness is independent of the "Keynesian" assumption of sticky prices.[25]

We, however, leave its detailed analysis to future occasions. Our search model is still too primitive to be able to provide a comprehensive guide for macroeconomic policy-making. We have to be content with just having laid out some elements of the foundation for macroeconomic analysis.

Appendix 1: The Proof of Existence, Uniqueness and Global Stability of Steady-State Fiat Money Equilibrium

The first purpose of this Appendix is to prove Proposition 3 of the main text. This can be reduced to the proof that the system of N(N-1) differential equations (7), together with the adding-up equation (6b), has a unique and globally stable steady-state $\{f^*_{ij}\}$ defined by (9), as long as the economy is connected and $0 < M < 1$.

Define a set of double indices $E = \{ij: e_{ij} > 0\}$. Since $f_{ij} = 0$ for all t whenever $e_{ij} = 0$, we only have to examine the dynamic motion of f_{ij} for ij ε E. (This will automatically exclude from E the double indices of the types of i0 and 0j.) To simplify the exposition, we sometimes rewrite (7) as:

$$\text{(A-1)} \quad \dot{f}_{ij} = F_{ij}(f) \quad \text{for } ij \ \varepsilon \ E ,$$

where f denotes a vector of all the f_{ij}'s with ij in E. It is then trivial to see that a time differentiation of the adding-up equation (6b) leads to the following equation:

$$\text{(A-2)} \quad \sum_{ij \varepsilon E} \sum F_{ij}(f) = 0 .$$

The task of the present Appendix is now reduced to that of proving the existence, uniqueness and global stability of steady-state vector f^* which satisfies $F_{ij}(f^*) = 0$ for all ij ε E. In fact, the proofs that will follow are in many ways analogous to those given in the standard Walrasian equilibrium model, which can be found, for instance, in Arrow and Hahn [1971]. Existence proof is quite straightforward.

(<u>a. Proof of the existence of a steady-state</u>): Consider a mapping of $f_{ij} + F_{ij}(f) \rightarrow f_{ij}$ for ij ε E. Since this maps a simplex, $\sum\sum_{ij \varepsilon E} f_{ij} = 1-M$ (>

0), continuously into itself, it has by Brower's fixed point theorem (see Arrow and Hahn [1971], App. C, or any textbook of real analysis) a fixed point f^* such that $f^*_{ij} + F_{ij}(f^*) = f^*_{ij}$ or $F_{ij}(f^*) = 0$ for all $ij \varepsilon E$. (QED)

In order to demonstrate the uniqueness and global stability of steady-state solution f^*, we need some Lemmas. First, we show that for any $ij \varepsilon E$ f_{ij} will eventually avoid the boundary values, 0 and e_{ij}.

<u><Lemma 1>: Suppose $0 < M < 1$. Then, for all $ij \varepsilon E$ there is a finite time T such that $0 < f_{ij}(t) < e_{ij}$ for any $t > T$.</u>

(Proof): Suppose $f_{ij} = 0$ at some point in time. Then, by (7) we have $df_{ij}/dt = (\Sigma_k f_{jk})e_{ij} \gtrsim 0$. Since $e_{ij} > 0$ for $ij \varepsilon E$, f_{ij} can remain zero only if $f_{jk} = 0$ for all $jk \varepsilon E$. But we now show that all the f_{jk}'s cannot remain zero forever. Because of the supposed inequality $1-M > 0$ the adding-up condition (6b) implies $\Sigma_h f_{k'h} > 0$ for at least one k', and we have by (7) $df_{jk'}/dt = (\Sigma_h f_{k'h})e_{jk'} > 0$ for such k'. Hence, $f_{jk'}$ will become positive in a finite period of time, thereby pushing f_{ij} above zero in a finite period of time as well. The eventuality of $e_{ij}-f_{ij} > 0$ can be proved in a similar fashion. Once all f_{ij}'s $(ij \varepsilon E)$ have entered the interior region, it is not difficult to show that they will never hit the boundaries again. (QED)

An immediate implication of this lemma is that $0 < f^*_{ij} < e_{ij}$ for any $ij \varepsilon E$.

Next, let us calculate the partial derivatives of $F_{ij}(f)$ for $ij \varepsilon E$.

(A-3a) $\partial F_{ij}/\partial f_{ij} = -\{\Sigma_k(e_{ki}-f_{ki}) + \Sigma_k f_{jk}\}$; (A-3b) $\partial F_{ij}/\partial f_{ji} = e_{ij}$;

(A-3c) $\partial F_{ij}/\partial f_{ki} = f_{ij}$ for $k \neq i$; (A-3d) $\partial F_{ij}/\partial f_{jh} = (e_{ij}-f_{ij})$ for $h \neq i$;

(A-3e) $\partial F_{ij}/\partial f_{kh} = 0$ for $k \neq i$ and $h \neq j$.

We have the following lemma which asserts the "indecomposability" of the set of these partial derivatives.

<Lemma 2>: Suppose that an economy is connected with $0 < M < L$. Then, there is no proper subset E' of E such that $\partial F_{ij}/\partial f_{ab} = 0$ for all $ij \in E'$ and $ab \notin E'$ at least for $t > T$.

(Proof): Suppose there is such a proper subset E'. Now, by the connectedness of the economy we can find in E a connected sequence of double indices bc, cd, ..., hi and ij such that $e_{bc} > 0$, $e_{cd} > 0$, ..., $e_{hi} > 0$ and $e_{ij} > 0$. Since we have by Lemma 1 $f_{bc} > 0$, $f_{cd} > 0$, ..., $f_{hi} > 0$ and $f_{ij} > 0$ at least for t after T, we have by (A-3) a connected sequence of positive partial derivatives: $\partial F_{bc}/\partial f_{ab} = f_{bc} > 0$, $\partial F_{cd}/\partial f_{bc} = f_{cd} > 0$, ..., $\partial F_{hi}/\partial f_{gh} = f_{hi} > 0$ and $\partial F_{ij}/\partial f_{hi} = f_{ij} > 0$. This immediately implies that ab, bc, cd,..., egh, hi and ij are all in E'. A contradiction. (QED)

We are now ready to prove the uniqueness and global stability of steady-state solution f^*. The key to the proofs given below is that the adding-up equation (A-2) plays a role analogous to Walras law and the system of F_{ij} whose partial derivatives are given by (A-3a) -- (A-3e) behaves like an excess demand system with weak gross substitutability and indecomposability (or connectedness) in the standard Walrasian theory.

(b. Proof of the uniqueness of steady-state): Let us denote by Φ a square matrix of all the partial derivatives $\partial F_{ij}(f^*)/\partial f_{kh}$ with both ij and kh in E except its first element 12. (If the original 12 is not in E, we just rearrange the indices appropriately.) According to the standard result on the uniqueness of Walrasian equilibrium (as is reviewed in Arrow and Hahn [1971], ch. 9), if all the principal minors of $-\Phi$ are shown to be positive, there can be only one steady-state f^* in our dynamical system. Now, we know from (A3-a) and Lemma 1 that $-\Phi$ has positive diagonal terms and semi-negative off-diagonal terms. Moreover, by Lemma 2 $-\Phi$ is also indecomposable. Hence, for a large enough positive number $s > 0$, we can construct a non-negative and indecomposable matrix $A = sI + \Phi$, where I is an identity matrix. To obtain a further property of $-\Phi$, let us differentiate the adding-up equation (A-2) with respect to f_{kh} for all kh in E except for hk = 12, and evaluate the resulting equations at f^*.

$$\text{(A-4)} \qquad -\sum_{ij \in E/12}\sum \frac{\partial F_{ij}(f^*)}{\partial f_{kh}} = \frac{\partial F_{12}(f^*)}{\partial f_{kh}} \quad \text{for each } kh \in E/12 .$$

Denote by 1 a vector of all 1's. Since by (A-3b)--(A3e) $\partial F_{12}(f^*)/\partial f_{kh}$ is non-negative for every kh $\in$ E/12 and since by Lemma 1 at least one of them is strictly positive, the above set of equations is in fact saying that there is a positive vector 1 such that

$$\text{(A-4')} \qquad -\Phi 1 = (sI - A)1 \geq 0 .$$

This is, however, precisely the condition for the positivity of all the principal minors of a matrix $-\Phi = (sI - A)$, when A is non-negative and indecomposable. (See, for instance, Theorem 4.D.2 of Takayama [1974].) (QED)

(c. Proof of the global stability of the steady-state): The global stability of f^* can be verified by the second Lyapounov method. (See Arrow and Hahn [1971], ch. 11 or any textbook of differential equations.) The Lyapounov function we shall introduce in the following is similar to the one used by McKenzie [1960]. Let $P = \{ij \in E: F_{ij}(f) \geqq 0\}$ and $N = \{ij \in E: F_{ij}(f) < 0\}$. They are of course disjoint. We then define a function:

$$\text{(A-5)} \qquad W(f) = \sum_{ij \in P}\sum F_{ij}(f).$$

Clearly, $W(f)$ is continuous, $W(f^*) = 0$ and $W(f) > 0$ for any $f \neq f^*$. If we can show that $W(f)$ is monotone decreasing for any $f \neq f^*$, this $W(f)$ is indeed a Lyapounov function. (Because of Lemma 1, we only have to examine its behavior for $0 < f_{ij} < e_{ij}$.) This in view, note first that by the adding-up equation (A-2) we can also write the above function as

(A-6) $W(f) = -\sum_{ij\in N}\sum F_{ij}(f)$.

Then, at a point where the derivative exists, we have from (A-6) and (A-7)

$$\dot{W}(f) = \sum_{ij\in P}\sum (\partial W/\partial f_{ij})\dot{f}_{ij} = \sum_{ij\in P}\sum \{-\sum_{kh\in N}\sum (\partial F_{kh}/\partial f_{ij})\}F_{ij} .$$

Since by (A-3b)--(A-3e) $\partial F_{kh}/\partial f_{ij} \geqq 0$ for any $kh\in N$ and $ij\in P$ and since by Lemma 2 $\partial F_{kh}/\partial f_{ij} > 0$ for some $kh\in N$ and $ij\in P$, we have indeed proved

(A-7) $\dot{W}(f) < 0$ at $f \neq f^*$.

The proof of monotone decreasingness of W at a point with no derivative is more involved but can be done just as in Lemma 4 of McKenzie [1960]. Hence, W(f) satisfies all the conditions for Lyapounov function, thereby demonstrating the global stability of f^*. (QED)

The second purpose of the present appendix is to prove Proposition 4.

(Proof of Proposition 4): In view of (8), (5a) and (5b), the existence, uniqueness and global stability of the steady-state supply-need frequencies $\{f^*_{ij}\}$ immediately imply the existence, uniqueness and global stability of $\{q^*_{oi}\}$ and $\{q^*_{io}\}$. Again in view of (8), (5a) and (5b), Lemma 1 implies the positivity of $\{q_{oi}\}$ and $\{q_{io}\}$ as long as for each i there is at least one k such that $e_{ki} > 0$ and at least one h such that $e_{ih} > 0$. The assumed connectedness of the economy clearly assures these inequalities. (QED)

Appendix 2: Welfare comparison among autarky equilibrium, barter equilibrium and fiat money equilibrium.

Let V^A_{ij}, V^B_{ij} and V^{FM}_{ij} respectively denote the representative individual's expected life-time utility in autarky, barter and fiat money equilibrium. In a state of autarky equilibrium we have $q_{ij} = 0$, in a state of barter equilibrium we have $q_{ij} = f_{ij} = e_{ij}$, and in a state of steady-state fiat money equilibrium we have $q^*_{i0} = \Sigma_{k\neq 0} f^*_{ik} = \Sigma_{k\neq 0}\{q^*_{0k}/(q^*_{k0}+q^*_{0i})\}e_{ik}$ and $q^*_{0j} = f^*_{0i} = \Sigma_{k\neq 0}\{q^*_{0k}/(q^*_{j0}+q^*_{0k})\}e_{ki}$ for all i ($\neq$ 0) and j ($\neq$ 0 and i). We thus have

(A-8a) $V^A_{ij} = -\infty$, (A-8b) $V^B_{ij} = u - b - c/e_{ij}$,

(A-8c) $$V^{FM}_{ij} = u-2b - \frac{c}{\Sigma_{k\neq 0}\{q^*_{0k}/(q^*_{k0}+q^*_{0i})\}e_{ik}} - \frac{c}{\Sigma_{k\neq 0}\{q^*_{0k}/(q^*_{k0}+q^*_{0i})\}e_{ik}} .$$

Obviously we have $V^A_{ij} < V^B_{ij}$ and $V^A_{ij} < V^{FM}_{ij}$. But it is in general not possible to compare V^B_{ij} and V^{FM}_{ij}. In the special case of doubly symmetric endowment-need distribution (13a), we have

(A-9a) $V^B_{ij} = u- b - cN(N-1)$;(A-9b) $V^{FM}_{ij} = u - 2b - cN/(1-M/p) -cN/(M/p)$.

Hence, $V^B_{ij} - V^{FM}_{ij} = b - cN\{N-1-1/\{(1-M/p)(M/p)\}\}$. This implies that as long as $N \leq 1+ 1/\{(1-M/p)(M/p)\}$ ($\geq$ 5) $V^B_{ij} > V^{FM}_{ij}$, but that for a sufficiently large N we can have $V^B_{ij} < V^{FM}_{ij}$.

In the second special case of minimally connected endowment-need distribution (13b), no barter equilibrium exists and we have $V^A_{ij} = V^B_{ij} = -\infty < V^{FM}_{ij} = u - 2b - cN/(1-M/p) -cN/(M/p)$.

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Footnotes

1. Menger [1871], Appendix J.

2. Clower [1967]. See Kohn [1988] for a helpful survey on the recent development of cash-in-advance constraint models.

3. Samuelson [1958]. See Kareken and Wallace [1980] as a useful collection of overlapping generations models.

4. See Jones [1976] for an earlier attempt to explain the phenmenon of money on the basis of search-theoretic model. See also Oh [1986].

5. In the present paper, the index 0 is reserved for the piece of state-issued paper to circulate as fiat money, and all the remaining indices are for "real" goods which are the objects of consumption for some individuals in the economy.

6. In one of the sequel papers we shall present an alternative model in which everybody searches for each other randomly in a single and huge trading zone.

7. $\{f_{ij}\}$ may not add up to one, because of the possibility of some individuals abstaining from entering the economy.

8. Without loss of generality we have chosen the unit of time in such a way that the meeting probability becomes exactly equal to the frequency q_{ij} per unit of time. Note that we have adopted the continuous time representation in order to make the dynamical analysis to be presented in section 5 simpler than the discrete-time representation we adopted in our previous paper.

9. See Diamond and Maskin [1979] and Mortensen [1982].

10. We shall examine the model with infinitely-lived individuals with time discounting in one of our sequel papers.

11. We adopt the following tie-breaking rule. When the strategies with different lengths of exchange sequence give the same value in (1), the one with the shortest length is chosen, and when the strategies with the same length of exchange sequence give the same value in (1), one of them is chosen randomly.

12. If we are willing to sacrifice the simplicity of exposition, it is possible to relax this assumption by replacing the autarky utility level by some finite number, say w. In this more general case, all the propositions in the present paper remain true if the utility of consumption u is suuficiently large.

13. There is also a trivial exchange equilibrium in which every individual stays autarky because every other individual stays autarky. It is not hard to show that such autarky equilibrium always exists in our search economy.

14. For the purpose of distinction, we shall henceforth relabel what we called

"monetary equilibrium" in our previous paper a "monetary equilibrium with commodity money" or simply a "commodity money equilibrium", because money circulated in it is a full-fledged commodity which is consumable by some private individuals in the economy. It is of some interest here to record the definitions given by Keynes to commodity money and fiat money. "Commodity money is composed of actual units of a particular freely obtainable, non-monopolized commodity which happens to have been chosen for the familiar purpose of money, but the supply of which is governed -- like that of any other commodity -- by scarcity and cost of production. Fiat money is representative (or token) money (i.e. something the intrinsic value of the material substance of which is divorced from its monetary face value) -- now generally made of paper except in the case of small denominations -- which is created and issued by the State, but is not convertible by law into anything other that itself, and has no fixed value in terms of an objective standard." (Keynes [1930].)

15. The so-called "state theory of money" is an attempt to ground the phenomenon of money on the legislative power of the state. See Lerner [1947].

16. In general, the state is able to increase or decrease the level of M by issuing new fiat money and buying some goods or by levying real taxes on people and selling the acquired goods.

17. To be precise, this assumption should be stated as $1/L \leq M \leq 1-1/L$, where L denotes the number of individuals in the economy.

18. If we sum these differential equations over all i ($\neq 0$) and j ($\neq 0$ and i), we can easily show that $\Sigma_{i \neq 0}\Sigma_{j \neq 0}\, df_{ij}/dt = 0$. This is another way of expressing the adding-up equation (6a) or (6b).

19. One natural form of such social welfare function is the following one: $V = \Sigma_{i \neq 0}\Sigma_{j \neq 0}\{e_{ij}/(1/q^*_{0i}+1/q^*_{j0})\}V^*_{ij}$, where $\{1/(1/q^*_{0i}+1/q^*_{j0})\}e_{ij}$ is the steady-state frequency of i-endowed, j-consuming individual and V^*_{ij} is their steady-state expecte life time uitlity.

20. Indeed, if the social welfare function has the following form: $V = \Sigma_i\Sigma_j(q^*_{0i}q^*_{j0})\cdot V^*_{ij}$, rather than the one supposed in footnote 12, its value can be calculated explicitly as $(u-2b)M(1-M)-cN^2$, and the optimum M becomes equal to 1/2, without having recourse to the special examples. The problem for this formulation, however, is that it is hard to give a reasonable justification for the weighting factor $q^*_{0i}q^*_{j0}$.

21. If we introduce an inventory process which allows each individual to adjust the amount of good or money to carry around in trading zones or if we introduce

a production process which allows each individual to determine the amount of good to bring to a trading zone, we may be able to incorporate some elements of decentralized price negotiations into our model.

22. It is the work of Diamond [1984] which has inspired the use of Nash bargaining solution in many of recent search-theoretic models. This solution can be shown to be an equilibrium outcome of the steady-state sequential barganing model of Rubinstein and Wolinsky [1985] in the limiting case of no time preferences.

23. See Proposition 7 and Proposition 8 in our previous paper [1988].

24. See Proposition 5 and Proposition 9 in our previous paper [1988]. In fact, in the case of minimally connected endowment-need distribution (13b), there is no barter equilibrium in the economy.

25. See Diamond [1984b] and Cooper and John [1988] for some illuminating discussions on the relationship between multiplicity of equilibria and potential effectiveness of macroeconomic policies.

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