This paper was published in
Structural Change and Economic Dynamics, 7(4)
Dec. 1996, pp. 451-477; “Corrigendum,” 9 1998, p. 269 .

The Bootstrap Theory of Money: A Search-Theoretic Foundation of Monetary Economics*

Katsuhito Iwai
Faculty of Economics, the University of Tokyo

May 1996
Corrected: July 1997

Abstract

The paper develops a search-theoretic model of decentralized economy with heterogeneous individuals trading goods endowed for goods in need among themselves. It characterizes both barter and monetary system as two different forms of trade equilibrium and demonstrates that while the existence of barter equilibrium requires a well-balanced distribution of endowments and needs, that of monetary equilibrium requires no "real" conditions. Indeed it argues that money is accepted as money by everybody merely because it is accepted as money by everybody else. The paper also points out, however, that there is a fundamental difficulty in the "natural" evolution of money and that in order for an economy to achieve a potential monetary equilibrium a large disturbance has to break the intrinsic symmetry of barter system.

JEL classification codes: E40, D83
Keywords: money, decentralized exchange, coordination, evolution, search

---

* This paper is a much shortened version of my earlier paper circulated under the title: "The Evolution of Money: A Search-theoretic Foundation of Monetary Economics," mimeograph, the University of Tokyo (December 1987) and CARESS Working Paper #88-03, University of Pennsylvania (February 1988). I am grateful to Messrs. Takashi Negishi, Masahiro Okuno-Fujiwara and Tadashi Shigoka for their helpful criticisms on its earlier versions. I have also benefited from the comments of the participants of my seminars at Tsukuba, Tokyo, Pennsylvania, Dartmouth, Yale, Harvard, Yokohama National and Ohio State. All the remaining errors are of course mine.

0. Introduction

Money is a medium which is accepted in exchange, not to be used directly for consumption or production, but to be exchanged for some other good with some other person at some other time. Money also serves as a measure of and a store of value, but these functions, though inseparable from money, can be and actually are shared by many other goods. Money is, however, not a mere medium of exchange; it is a "general" medium of exchange which is accepted by anybody at any place at any time. It is thus able to overcome the difficulties of barter trade which requires a "double coincidence of wants" -- a situation in which one individual has the goods that the other individual needs and needs the goods that the other individual has.[1]

That the existence of a general medium facilitates the decentralized exchange process is of course a matter of elementary economics. But why and how a certain good is accepted as a general medium of exchange is another matter. Indeed, it was more than a hundred years ago that Carl Menger wrote the following in his classic work on money.

> The great thinkers of antiquity, and following them a long series of the most eminent scholars of later times up to the present day, have been more concerned than with any other problem of our science with the explanation of the strange fact that a number of goods (gold and silver in the form of coin, as civilization develops) are readily accepted by everyone in exchange for all other goods, even by persons who have no direct requirements for them or whose requirements have already been fully met. (Menger [1871])

The solution Menger proposed for this problem is what he called a theory of the "salability" of goods. A good has high salability if its "possession would considerably facilitate the individual search for persons who have just the goods he needs". However, not all goods are equally salable. While there is a limited demand for certain goods, that for others may be very general. And when an individual has goods with low salability, it is often difficult to obtain the goods he needs by direct barter. He may find it more economical to exchange his own goods first for a more salable good even if he himself does not need it and use the latter as a medium for obtaining the

[1]The best-known work on this problem is of course W. S. Jevons [1875]; more recent contributions include Starr [1972], Ostroy and Starr [1974], and Niehans [1978].

goods he really needs in later times. Menger then claimed that "as each economizing individual becomes increasingly more aware of his economic interest, he is led by this interest, without any agreement, without legislative compulsion, and even without regard to the public interest, to give his commodities in exchange for the other, more salable, commodities, even if he does not need them for any immediate consumption purpose," and concluded that "with economic progress, we can everywhere observe the phenomena of a certain number of goods ... becoming acceptable to everyone in trade," that is, becoming money.[2] The aim of Mengerian theory of money was to formalize the Invisible Hand explanation of the evolution of money given in the famous chapter on the origin and use of money in *The Wealth of Nations*.[3]

The present paper is an attempt to develop a simple search-theoretic model of decentralized exchange economy, which is capable of analyzing both barter and monetary trade system as two of many possible forms of trade equilibrium.[4] Its chief objective is to demonstrate that while the existence of barter equilibrium requires a well-balanced distribution of endowments and needs among individuals, that of monetary equilibria requires only what we call the "connectedness" of an economy -- the minimum requirement that a collection of individuals forms at least potentially a self-sufficient economy. Indeed, we first confirm the Mengerian logic within our formal search model by showing that when there is a good with high salability (as well as with what we will call

---

[2] Menger [1871]; see also his well-known paper [1892] on the origin of money.
[3] Simth [1776], Book I, chapter IV "Of the Origin and Use of Money."
[4] See Jones [1976] for a pioneering attempt to explain the medium of exchange function of money on the basis of search-theoretic model of decentralized exchange process. His model assumed that each individual commits oneself to a simple (and in general suboptimal) trade strategy and meets with each other randomly in a large differentiated trading zone. His model also assumed that the same fraction of traders wishes to buy each good as wishes to sell, or to use our notation in section 1, that $q_{ij} = p_i p_j$. The present paper follows the lead of Jones and solves his model completely. First, it replaces the Jonesian single trading zone by multiple trading zones each of which is specialized to the bilateral trades between a prespecified pair of goods and lets each individual choose the sequence of the zones to visit optimally. Second, it is able to drop the assumption of $q_{ij} = p_i p_j$. Since the original version of this paper was written, there appeared two important papers which also followed the lead of Jones. Oh [1989] replaced the pre-committed strategies of Jonesian traders by optimal sequential strategies. Kiyotaki and Wright [1989] also reframed the Jones model in a sequential manner and added production and storage costs to it. They gave a complete characterization of trade equilibria, including the one with commodity money and another with fiat money, but only in the case of three different goods being produced and traded by three types of individuals with equal population.

high purchasability) everybody finds it less costly to use it as a general medium of exchange than to seek a barter trade. But the far more fundamental for our claim of the complete universality of monetary equilibrium is an observation that this Mengerian logic is causally reversible. In fact, we are going to show that once a certain good has come to be accepted as a general medium of exchange the very use of it as a general medium of exchange endows it with the maximum salability (as well as the maximum purchasability) at the expense of all the other goods in the economy, thereby creating the very conditions for its own acceptance as a general medium of exchange. Money is money simply because it is used as money. It is the inherent increasing-returns-to-scale nature of matching technology that works to reverse the causal order. What Menger called the salability of goods turns out to be not an exogenous parameter but an endogenous variable whose magnitude is affected by the very trading structure of the economy.

Difficulties of barter trade arise from its being constrained by the "real" structure of the economy. The double coincidence of wants is the condition for the way "technology and preferences" are distributed among different individuals. Money, on the other hand, is a "social entity" which is capable of sustaining itself by its own bootstraps. And it is because money requires no "real" foundations to support itself that it is able to overcome the "real" constraints of the economy and make the otherwise impossible decentralized exchanges possible.

Money is thus a potentially ubiquitous entity. But, as any student of the speculative philosophy knows, there is a wide gap between the potentiality and the actuality, and, however tempting it is, we cannot immediately jump to the assertion that money evolves "naturally" in any economy. The logic of money should not be confused with the genesis of money. In fact, the second objective of the present paper is to point out the fundamental difficulty in the evolution of money and monetary economy. While the system of barter trades treats all goods symmetrically, that of monetary trades creates an artificial asymmetry by assigning a social role to an arbitrarily chosen good or, in the case of fiat money, to a fictitious good. Unless some outside enforcement or some historical accident or some other form of shocks and disturbances were to break the natural symmetry of barter trades, the bootstrap mechanism which is to sustain the circulation of one particular good as the general medium of exchange would never be set in motion. (And, once an economy has settled down to one of its potential monetary equilibria, it becomes difficult to stop and reverse the on-going bootstrap mechanism. There is a certain irreversibility in the evolutionary process of money.) In contradistinction to the claim of Carl Menger, even if "each economizing individual becomes increasingly aware of his economic interest," money may never evolve "naturally" from the system of barter trades. We need something more than the Invisible Hand to explain the evolution of money and monetary economy.

But we seem to have anticipated too much of what follows. We better start presenting our search-theoretic model of barter and monetary trade processes at once.

1. The basic search-theoretic model of decentralized trade process.

Consider an economy with N goods and a large number of heterogeneous individuals. Each individual enters the economy with an endowment of one unit of one good and with a need to consume another good. The good endowed and the good in need are both assumed to be fixed *a priori*. (This presupposes an extreme form of the division of labor.) We denote the frequency (realtive to the total population) of those individuals with an endowment of good i and with a need of good j by $\bar{e}_{ij}$, and call it the "endowment-need frequency". (Note that $\bar{e}_{ii} = 0$ by assumption and $\sum_i\sum_j \bar{e}_{ij} = 1$ by construction.) To simplify the later exposition, we assume that, as soon as an i-endowed j-consumer retires from the economy to consume good j, a new i-endowed j-consumer enters into it with a new unit of good i. Such instantaneous parent-child succession keeps $\bar{e}_{ij}$ constant over time, irrespective of the way individuals trade with each other.[5]

It should be noted that the set of $\bar{e}_{ij}$'s represents the "real" data of our exchange economy which summarizes its "technology and preferences" or what may be called the "fundamentals" of the economy. We introduce:

Assumption: An economy is "connected" in the sense that for any i and j (≠ i) we have a connected sequence of positive endowment-need frequencies such that $\bar{e}_{hi} > 0$, $\bar{e}_{gh} > 0$, ..., $\bar{e}_{kl} > 0$ and $\bar{e}_{jk} > 0$.[6] ◊

---

[5] This assumption of the constancy of eij becomes very natural in the case of infinitely-lived individuals, discussed in footnote 9.
[6] The notion of connectedness given above is closely related to that of "irreducibility" in the theory of Markov chains. See Feller [1968].

If an economy is connected, an almighty authority watching over it could satisfy the real need of any of its members by ordering him to give his initial endowment to a second individual in need of it, who is in turn ordered to give his initial endowment to a third individual in need of it, and so on, until the sequence of orders reaches an individual whose initial endowment is the very good the first individual is in need of. As long as the economy is connected, all of its members could in principle be liftted from the misery of autarky by a centralized trade coordination. The connectedness is thus the minimum requirement for an association of individuals to form "a" self-contained economy, not a mere collection of isolated individuals and/or disjoint communities.

The fundamental question we now pose here is whether such a trade coordination is possible in a decentralized manner, that is, without the benevolent intervention of an almighty authority. In order to answer this question, we have to develop a model of decentralized trades.

Before we start our analysis, let us keep in record two special examples of the connected economy which will serve as useful benchmarks in our later analysis. The first is the economy with a doubly symmetric endowment-need distribution, or

(1a) $\bar{e}_{ij} = \bar{e}_{ji} = 1/N(N-1)$ for any i and j ($\neq$ i).

This is of course the most densely connected economy. The second is the economy with a minimally connected endowment-need distribution, or

(2b) $\bar{e}_{12} = \bar{e}_{23} = ... = \bar{e}_{N-1,N} = \bar{e}_{N1} = 1/N$; and all the other $\bar{e}_{ij} = 0$.[7]

We now have to describe the way different individuals meet each other and trade each other in a decentralized economy. We suppose that there are N(N-1)/2 separate trading zones and that each trading zone is specialized to trades between a given pair of goods (i,j). (Hence, (i,j) zone and (j,i) zone are identical.) Since we have purged any centralized trade coordinator from our economy, it is the task of each individual to decide a trading zone to visit in order to find a trading

[7] This barely connected economy is in fact the example Cass and Yaari [1966] used as a static counterpart of the well-known consumption-loan model of Paul Samuelson [1958]. In fact, our model of decentralized exchange economy has a certain similarity to consumption-loan models. There are, however, at least two distinguishing features. First, while the consumption-loan models presuppose the use of some piece of paper as fiat money from the beginning, our model is able to explain that possibility endogenously. Secondly, while the medium of exchange function and the store of value function of money are inextricably intermingled in consumption-loan models, it is the medium of exchange function of money which is put in full relief in our model.

partner in it. For instance, an i-endowed j-consumer may decide to make a direct trip to (i,j) zone and search for a mirror-symmetric individual who is willing to trade good j for good i. This is the barter strategy. The same purpose can also be accomplished in a roundabout way. The same individual may first make a trip to (i,k) zone (with $k \neq j$) and search for another individual willing to trade good k for good i. Having found such a partner and obtained good k, he then makes a second trip to (k,j) zone and search for another individual willing to trade good j for good k. This is the strategy of indirect trade which uses good k as a "medium of exchange". Indirect trade strategies using more than one media of exchange are also possible.[8]

We still need a description of the matching technology in each trading zone. In (i,j) trading zone two types of individuals are searching each other -- those who want to supply good i in return of good j, and those who want to supply good j in return of good i. Denote by q_{ij} the frequency (relative to the total population) of those i-supplying j-demanders and by q_{ji} the frequency of those j-supplying i-demanders, and call them the "supply-demand frequencies." ($q_{ii} = 0$ by assumption and $\sum_i\sum_j q_{ij} = 1$ by construction.) If each trading zone is large and is populated sparcely by searchers (and this is the reason for our having called it a "trading zone" rather than a "trading post"), it is reasonable to assume that the probability of encountering one of the i-supplying j-demanders in (i,j) zone per unit of time is proportional to their frequency q_{ij}.[9] (In other words, the probability of an encounter is subject to a Poisson process.) Since we have a degree of freedom in choosing a time unit, we can fix this probability equal to q_{ij}. It is important

[8] Perhaps the simplest way to formalize the decentralized trading structure is to adopt a biological mate-matching model by assuming that people search each other randomly in a large unspecified trading zone. But we have not adopted this seemingly "natural" assumption because rational human-beings have various means to communicate their intentions prior to their actual trades. If for example everyone carries a placard indicating what he is ready to demand and supply, much of unfruitful encounters can be avoided. In fact, our multiple zone model is formally equivalent to this placard model. Moreover, by imposing some pre-trading restriction on trading processes, we are able to obtain much sharper (and perhaps the sharpest) results than in the random search model.

[9] As a matter of fact, we can generalize this assumption in such a way that the probability of meeting an i-supplying j-demander is proportional to $P(q_{ij})$, where $P(0) = 0$ and $dP(q)/dq > 0$ for all $q > 0$, without losing any of the propositions that will follow. In fact, this generalization would enable us to deal with the trading "post" model in which the probability that an j-supplying i-demander meets an i-supplying j-demander can be written as $\mathrm{Min}[q_{ij}/q_{ji}, 1]$. Because of the space limit, we shall not study this trading post model in the present paper.

to note that this implies that the matching technology is subject to increasing returns to scale, because the aggregate number of meetings in each zone increases with the order of the square of the frequency of searchers in it.[10]

The notion of supply-demand frequency q_{ij} captures at least a part of what Carl Menger called "salability" of goods. It is because the possession of a good, say m, with high q_{im} would considerably facilitate the individual search for another individual who is willing to supply the good i he needs. Likewise, a good m with high q_{mj} may be said to have high "purchasability", though Menger himself did not use such an notion. It is because its possession would considerably facilitate the individual's search for another individual who needs the good j he is willing to supply.

We assume that everyone has correct expectations, so that his subjective estimate of q_{ij} is identical with its objective value. We, however, relax this assumption in section 9 in order to study the problem of the evolution of money, i.e., the problem of whether dynamic interactions between people's subjective beliefs about the salability and purchasability of different goods and the actual salability and purchasability determined by the very trading process among people would, as was insisted by Menger, produce a "natural" tendency to the general use of a medium of exchange. We also suppose that people can carry only one unit of one good in their inventory and that the exchange ratio between any two goods is fixed at one-to-one. Though the problem of price-formation becomes important once a general medium of exchange starts circulating in the economy and splits the unity of barter trade into a sale and a purchase, we do not believe it crucial for our understanding of the essential nature of money.

2. Individual trade strategy.

---

[10] Indeed this corresponds to what Diamond and Maskin [1979] called the "quadratic meeting technology" in their search model. See also Mortensen [1982]. Diamond later applied this quadratic meeting technology model to the analysis of barter and monetary exchange processes in [1982] and [1984]. The prototype exchange model of Diamond, however, is a single-good barter exchange, and his "monetary economy" model presupposes a given structure of monetary transactions technology. It is one of the purposes of the present paper to deduce the very structure of the monetary transactions technology endogenously on the basis of the search-theoretic analysis of individual exchange behaviors.

Let us examine the optimal trade strategy of an i-endowed j-consumer in the decentralized economy described above. Denote by u the utility of consuming a unit of the needful good, by b the cost of engaging in a transaction with another, and by c the cost of spending a period in search. These parameters are assumed to be constant over time and uniform across both individuals and goods, in order to assure that no good is predestined to become money. (We also assume that the utility of not consuming the needful good, i.e., of staying autarky is $-\infty$.) We then suppose that each individual maximizes the undiscounted expected utility of his life-time which begins with the entry into the economy with an endowed good in hand and ends with the retirement with a needful good in hand.[11]

We denote the maximum life-time expected utility of an i-endowed j-consumer by V_{ij}. Its value can be calculated as follows. Suppose that this individual has decided to visit (i,k) zone first to search an individual who is willing to trade good k for good i. Since the probability of meeting such a trade partner in each time unit is q_{ki} (not q_{ik}!), the expected search period can be calculated as $1/q_{ki}$ and the expected search cost as c/q_{ki}, so long as q_{ki} is constant over time. As soon as a trading partner is found, a unit of good i is traded for a unit of good k at the expense of a transaction cost b. If k = j, this individual retires from the economy to enjoy the utility u of consuming good j. The life-time expected utility is then equal to $u-b-c/q_{ji}$. If, however, $k \neq j$, a search process will be started anew with good k in hand. Since u, b and c are all assumed to be uniform across individuals and goods, this individual now faces the same trading opportunities as

---

[11] In fact, we can transform our search model with finite-horizon individuals into that with infinite-horizon individuals by a mere reinterpretation of parameters in the following manner. Suppose that each individual has an ability to produce a unit of one particular good only after he has consumed a unit of another good and forever repeats an activity cycle of production-search&trade-consumption. If he maximizes the expected value of infinite-horizon discounted utility integral, then a given discount rate r (> 0) determines the search cost as an opportunity cost. Suppose also that each transaction takes time with a given completion probability w (> 0) per unit of time. The expected transaction period 1/w determines the transaction cost again as an opportunity cost. (We can easily introduce a production cost as well.) Then, using the dynamic programming argument similar to the one to be given in the text, the maximum expected discounted utility integral U_{ij} of an infinite-horizon i-endowed j-consumer can be calculated as $U_{ij} = \text{Max}_k\, [U_{kj}/\{(1+r/w)(1+r/q_{ki})\}]$. If this i-endowed j-consumer engages in *n* exchanges in each cycle, we then have $U_{ij} = \text{Max}_{k,..,h}\, [u/\{(1+r/w)^n(1+r/q_{ki})\ldots(1+r/q_{jh})-1\}]$. Since to maximize an expression $u/(L-1)$ is generally equivalent to maximize $\log(u)-\log(L)$, It can easily be shown that to maximize the R-H-S of this expression we identify $\log U_{ij}$, $\log(1+r/w)$ and $\log(1+r/q_{ki})$ respectively with V_{ij}, b and c/q_{ki}, this formula becomes exactly the same as (2) of Proposition 2.

a newly born k-endowed j-consumer. The expected utility from the search process to start from that instant is therefore equal to V_{kj}. The life-time expected utility can then be calculated as V_{kj}-b-c/q_{ki}. Moreover, if we adopt an obvious convention: V_{jj} = u, the same expression can subsume the expected utility of the barter strategy u-b-c/q_{ji}. We thus obtain:

Proposition 1 : If demand-supply frequencies $\{q_{ij}\}$ remain stationary, the maximum expected utility of an i-endowed j-consumer is characterized by the following functional equation:

(3) $$V_{ij} = \underset{k}{\mathrm{Max}}\ [V_{kj} - b - c/q_{ki}]\ .\quad \Diamond$$

Note in passing that since there are only finite number of possible strategies, the existence of an optimal search strategy is trivially guaranteed.

There is however a more leisurely way of calculating V_{ij}. Denote by $V^0{}_{ij}$, $V^1{}_{ij}$, $V^2{}_{ij}$, ... the maximum expected utility of an i-endowed j-consumer who has committed respectively to a barter trade, to an indirect trade using one medium, to an indirect trade using two media, and so forth. Then, the dynamic programming argument immediately leads us to the following set of equations.

(4a) $$V^0{}_{ij} = u - b - c/q_{ji}\ ,$$

(5b) $$V^1{}_{ij} = \underset{k}{\mathrm{Max}}\ [V^0{}_{kj} - b - c/q_{ki}\ ] = u\text{-}b - \underset{k}{\mathrm{Min}}\,[b + c/q_{jk} + c/q_{ki}]\ ,$$

(6c) $$V^2{}_{ij} = \underset{k}{\mathrm{Max}}\ [V^1{}_{kj} - b - c/q_{ki}\ ] = u\text{-}b - \underset{k,\,h}{\mathrm{Max}}\,[2b + c/q_{jh} + c/q_{hk} + c/q_{ki}]\ ,$$

and so forth. We can then express V_{ij} as the maximum of maxima, or

(7) $$V_{ij} = \underset{n}{\mathrm{Max}}\ [V^n{}_{ij}]\ .$$

To be complete, we need to specify a tie-breaking rule to choose among the trade strategies with the same expected utility. The rule we shall adopt is a lexicographic-cum-randomizing one which chooses the strategy with the shorter trade sequence when they have different lengths and tosses a coin when they have the same length.

3. Conditions for universal barter trade and universal monetary trade.

The structure of decentralized trades among heterogeneous individuals is in general very complex. But there are at least two trade structures which are simple or at least familiar. They are of course barter trade and monetary trade. We take up the barter trade first.

It is tautological to say that a barter strategy is chosen if it guarantees a utility higher than that of any indirect trade. This statement can be formalized compactly as $V_{ij} = u-b-c/q_{ji}$ or more intuitively as $V^0_{ij} \geq V^n_{ij}$ for any $n > 0$. Hence, we have from (8a), (9b), (10c), ...

(5a) $c/q_{ji} \leq b + c/q_{jk} + c/q_{ki}$ for any k ,

(5b) $c/q_{ji} \leq 2b + c/q_{jh} + c/q_{hk} + c/q_{ki}$ for any k and h ,

and so forth. Inequality (5a) says that the barter trade is at most as costly as any one medium indirect trade, (5b) that the barter is at most as costly as any two media indirect trade, and so forth. It, however, turns out that if (5a) holds for any individual type, i.e. for any i and j, all the longer inequalities, (5b) ... become redundant as sufficient conditions for the universal barter trade. For by repeatedly applying (5a) we have: $c/q_{ji} \leq b + c/q_{jk} + c/q_{ki} \leq 2b + c/q_{jk} + c/q_{kh} + c/q_{hi} \leq \ldots$. (We have used here the assumption of the uniformity of b and c.) Since the necessity of (5a) is obvious, we have established:

Proposition 2: Everyone chooses to barter if and only if the following inequalities hold for any i, j (≠ i) and k (≠ i and j):

(6) $c/q_{ji} \leq b + c/q_{jk} + c/q_{ki}$. ◊

At the other extreme lies a trade structure in which one particular good is used as the sole medium of exchange by everybody in the economy, except by the one who already has it and the one who really needs it. That good is now functioning as the general medium of exchange, i.e., as "money". Let this particular good be indexed by m. The question we now ask is: under what conditions does everyone come to use good m as money voluntarily, even if it would require him to visit two trading zones and engage in two transaction activities? The following proposition characterizes such conditions.

Proposition 3: A good m is used as money by everybody in the economy, except by the one who was born with it and by the one who really needs it, if and only if the set of supply-demand frequencies $\{q_{ij}\}$ satisfies the following set of inequalities;

(7a) $b + c/q_{jm} + c/q_{mi} < c/q_{ji}$ for any i (≠m) and j (≠ m and i);

(7b) $b + c/q_{jm} + c/q_{mi} < b + c/q_{jk} + c/q_{ki}$ for any i (≠ m), j (≠ m and i) and k (≠ m, i and j);

(7c) $c/q_{jm} \leq b + c/q_{jk} + c/q_{km}$ for any j (≠ m) and k (≠ m and j); and

(7d) $c/q_{mi} \leq b + c/q_{mk} + c/q_{ki}$ for any i (≠ m and j) and k (≠ m and i). ◊

The inequality (7a) says that it is less costly to use the good m as a medium of exchange than to barter directly. (7b) says that among all the possible media the good m is the least costly to use and no other good can rival it. (7c) then says that when one already has the good m (either by endowment or by exchange) it is less costly to trade it directly for the needful good than to use some other good as a medium. And (7d) says that when one has a need to consume the good m it is less costly to seek a direct barter than to use some other good as a medium. The economic interpretation of these inequalities is this simple, but the mathematical proof of their necessity and sufficiency is somewhat long and has been relegated to Appendix below.

In any case, we have just confirmed the logic of the Mengerian theory of money. A good m is used as money if and only if its salability q_{im} and purchasability q_{mj} are uniformly higher than those of all the other goods q_{ij} ($i \neq m$ and $j \neq m$). We will, however, see soon that this Mengerian logic covers only one half of the search theory of money.

4. Trade equilibrium.

We have so far analyzed the individual trade strategy as if supply-demand frequencies $\{q_{ij}\}$ are given exogenously. They are, however, not the "fundamentals" of the economy, and their actual values are determined by the very individual search and trade activities in various trading zones. We therefore have to relate these surface frequencies to the true "fundamentals" $\{\bar{e}_{ij}\}$. But their relationship turns out to be complex and dynamic, and we proceed step by step.

In our economy every individual is born with a fixed endowment of one good and a fixed need to consume another good. But, except for the barter strategy, the good endowed is bound to be traded for some medium of exchange, and it is the good currently held and not the good originally endowed that is relevant for the individual trade strategy. Accordingly, we denote by e_{ihj} the frequency (relative to the whole population) of the i-endowed j-consumers who are currently holding good h, and call it the "transient frequency". (Note that $\sum_h e_{ihj} = \bar{e}_{ij}$ and $e_{ijj} = 0$ by construction.)

Before us are three sets of frequencies -- $\{q_{ij}\}$, $\{e_{ihj}\}$ and $\{\bar{e}_{ij}\}$ -- each representing a different layer of the economy. The surface layer is the set of supply-demand frequencies $\{q_{ij}\}$ which are observable daily in trade zones, and the deepest layer consists of the set of endowment-need frequencies $\{\bar{e}_{ij}\}$ which summarizes the "fundamentals" of the economy. The set of transient frequencies $\{e_{ihj}\}$ just introduced serves as a bridge between these two frequencies. While $\{\bar{e}_{ij}\}$ is constant over time, both $\{e_{ihj}\}$ and $\{q_{ij}\}$ are subject to evolutionary changes through decentralized trading processes. We will first examine the relationship between $\{q_{ij}\}$ and $\{e_{ihj}\}$, postponing the study of the relationship between $\{q_{ij}\}$ and $\{\bar{e}_{ij}\}$ until section 7.

In fact, we can "count" each of $\{q_{ij}\}$ in terms of $\{e_{ihj}\}$ as follows. When i-endowed, i-holding, j-consumers have decided to barter good i for good j, they visit (i,j) zone and add their frequency e_{iij} to q_{ij}. When h-endowed, i-holding, k-consumers (with i $\neq$ k) have decided to obtain good j ($\neq$ i), they also join (i,j) zone, adding their frequency e_{hik} to q_{ij}. Now the condition that i-endowed, i-holding, j-consumers barter good i for good j can be written compactly as V_{ij} = u-b-c/q_{ji} and the condition that h-endowed, i-holding, k-consumers demand good j can be written compactly as $V_{ik} = V_{jk} - b - c/q_{ji}$. Taking note the convention $u = V_{jj}$, we are able to represent q_{ij} formally as:

(8) $$q_{ij} = \Sigma_h \Sigma_{\{k:\ Vik = Vjk\text{-}b\text{-}c/qji\}} e_{hik}$$ for any i and j ($\neq$ i) .[12]

We have now closed a full circle. Given a set of supply-demand frequencies $\{q_{ij}\}$, the functional equation (2) of Proposition 1 allows us to calculate the expected utility V_{ij} of any individual whose trade strategy is determined by its maximization. Then, the counting equation (8) just written down in turn determines these frequencies $\{q_{ij}\}$ as the very aggregate outcomes of these individual behaviors. We are thus in a position to introduce the formal definition of trade equilibrium of our decentralized exchange economy:

Definition: An economy is said to be in a state of "trade equilibrium" if its supply-demand frequencies $\{q_{ij}\}$ satisfy both (2) and (8) for any i and j ($\neq$ i) and none of its members are in a state of autarky. ◊

---

[12] To be rigorous, the expression in the second Σ in (8) is true only when V_{jk}-b-c/q_{ji} is the unique maximizer of V_{ik}. In general q_{ij} is a set-valued variable, and ehik in Σ should be replaced by (s^jik)ehik where s^jik takes any value in [0,1] with $\Sigma_g(s^g{}_{ik}) = 1$ where g = argmax$_j$ [V_{jk}-b-c/q_{ji}] . We ignore this complication in what follows.

It is one thing to define the notion of equilibrium, but it is another to analyze it. There are indeed a large number of possible forms of trade equilibrium, every one of which deserves a special attention. But the space limitation forces us to concentrate on the following two forms.

Definition: A trade equilibrium is called a "barter equilibrium" when every active individual barters with each other, and a "monetary equilibrium" when every acitive individual uses one particular good as the exclusive medium of exchange, except the one who is endowed with it and the one who is in need of it. ◊

5. The difficulty of barter equilibrium.

Since the barter equilibrium has a much simpler structure, we examine it first. Suppose that everybody seeks a barter partner. Then, the good he supplies is the good he is endowed with and the good he demands is the good he is in need of. We therefore obtain the following equality, without going through the counting equation (8).

(9) $q_{ij} = \bar{e}_{ij}$ for any i and j .

This equality immediately leads to a necessary condition for the existence of barter equilibrium -- a condition which should look familiar to every student of monetary economics.

Proposition 4: No barter equilibrium exists unless $\bar{e}_{ji} > 0$ whenever $\bar{e}_{ij} > 0$. ◊

(Proof): Suppose that $\bar{e}_{ji} = 0$ for some i and j such that $\bar{e}_{ij} > 0$. Then by (9) $q_{ji} = 0$, and we have $V_{ij} = u\text{-}b\text{-}c/q_{ji} = -\infty$, implying that i-endowed, j-consumers end up with a state of autarky. (QED)

This is of course the formalization of what Jevons called the "double coincidence of wants" for barter trades.[13] An economy cannot attain a barter equilibrium unless every individual is endowed with a good that some other individual needs to consume and is in need of the good that the same individual is endowed with. It is easy to see that the minimally connected economy (1b) lacks this condition and has no barter equilibrium. Even if everybody in a connected economy are potentially capable of satisfying their real needs by a suitable trade coordination, the barter form of decentralized trades may forever confine them to the shackle of autarky.

[13] See, for instance, Jevons [1875].

The double coincidence of wants is, however, only a necessary condition for the barter equilibrium. To understand its nature more fully, let us suppose that all individual types are active in the sense that $\bar{e}_{ij} > 0$ for any i and j ($\neq$ i). Then, Proposition 2, which stipulates the range of $\{q_{ij}\}$ that induces everybody to seek a barter partner, becomes directly applicable. Hence, the equality between q_{ij} and $\bar{e}_{ij}$ given by (9) immediately allows us to obtain:

Proposition 5: Suppose $\bar{e}_{ij} > 0$ for any i and j ($\neq$ i). Then, there exists a barter equilibrium if and only if the following inequality holds true for any i, j ($\neq$ i) and k ($\neq$ i and j):

(10) $c/\bar{e}_{ji} \leq b + c/\bar{e}_{ki} + c/\bar{e}_{jk}$. ◊

When $\bar{e}_{ij} \geq c/b$ for all i and j ($\neq$ i) a barter equilibrium always exists. This is the case where one's mirror-symmetric individual is easily found and no one in the economy bothers to seek an indirect trade. However, since $\Sigma_i \Sigma_j \bar{e}_{ij} = 1$, this case disappears as soon as the number of goods N becomes large and N(N-1) > b/c. More interesting is the case of doubly symmetric economy (1a). Since all $\bar{e}_{ij}$'s are equal, it is trivial to see that the inequalities (10) are satisfied and hence a barter equilibrium exists. Indeed, economies with more or less balanced endowment-need distribution tend to have a room for a barter equilibrium. But, as the distribution becomes more and more unbalanced, the inequality (10) becomes more and more likely to be violated. We already know that the minimally connected economy (1b) has no barter equilibrium, but we do not have to go to that extreme to observe the barter equilibrium collapse. Economies without barter equilibrium are not the exceptions but the rules.

6. The bootstrap nature of monetary equilibrium.

Let us turn to the analysis of "monetary equilibrium" -- a form of trade equilibrium which uses one particular good as the general medium of exchange, i.e. as money. Suppose that by some reason everybody in the economy, except the one endowed with it and the one in need of it, uses a good m as a sole medium of exchange. Then, the counting equation (8) can be rewritten in the following simpler manner.

(11a) $q_{im} = \Sigma_k e_{iik}$ for any i ($\neq$ m) ;

(11b) $q_{mj} = \Sigma_h e_{hmj}$ for any j ($\neq$ m) ;

(11c) $q_{ij} = 0$ for i $\neq$ m and j $\neq$ m .

What (11a) says is that those individuals who are endowed with any of the non-monetary goods always demand a monetized good first; (11b) that only those individuals who currently hold the monetized good demand the good they need; and (11c) that no current holders of a non-monetary good demand another non-monetary good. In fact, all the demanders for the monetized good who currently hold a non-monetary good i now become what we usually call the "sellers" of good i and all demanders for a non-monetary good j who currently hold the monetized good the "buyers" of good j. All together, the equations (11b), (11a) and (11c) in that order are nothing but the formal restatement of the well-known dictum of Robert Clower [1967] that "money buys good, goods buy money, but goods do not buy goods."

What is remarkable about these "Clower equations" of a monetary equilibrium is that they are "self-enforcing" in the sense that they give rise to the very conditions for the existence of a monetary equilibrium![14] The following proposition gives us a preliminary result.

Proposition 6: Suppose $\Sigma_k e_{iik} > 0$ for every i ($\neq$ m) and $\Sigma_h e_{hmj} > 0$ for every j ($\neq$ m). Then, an economy has a monetary equilibrium with good m used as money. ◊

(Proof): Suppose that good m is by some reason used as money. Then, by the Clower equations (11a) - (11c) we have $q_{im} = \Sigma_k e_{iik} > 0$ for any i ($\neq$ m), $q_{mj} = \Sigma_h e_{hmj} > 0$ for any j ($\neq$ m), and $q_{ij} = 0$ for i ($\neq$ m) and j ($\neq$ m). It follows trivially that the inequalities (7a) - (7d) in Proposition 3 are all satisfied, because while their L-H-S's remain finite their R-H-S's all become infinite. (QED)

Though formally trivial, what underlies the above Proposition is the working of a "bootstrap mechanism". Because of the increasing returns to scale nature of the matching technology, the more people use a particular good as a medium of exchange, the smaller the cost of using it as such and the higher the cost of seeking a barter trade as well as the cost of using another good as a medium. Hence, everybody in the economy (except the one endowed with it or the one in need of it) is induced to use that good as the sole medium of exchange. Money is supporting itself by its own "bootstraps".

[14] This perhaps answers the criticism of Frank Hahn [1981] that "the Clower procedure assumes what should be explained," p. 21.

Nonetheless the above proposition still falls short of establishing the bootstrap nature of money in its fullest sense. For the existence condition in it still requires the positivity of both the "purchasability" $q_{mj} = \Sigma_h e_{hmi}$ and the "salability" $q_{im} = \Sigma_k e_{iik}$ of the monetized good m against all the other goods i ($\neq$ m) in the economy. Does this mean that money still needs some kind of "real" foundations for its circulation as money? The answer is, however, No. But to justify this negative answer, we now have to look at the long-run relationship between $\{q_{ij}\}$ and $\{\bar{e}_{ij}\}$ by tracing out the dynamic evolution of the transient frequencies $\{e_{ihj}\}$.

7. The universal existence of monetary equilibria in the long-run.

Suppose again that by some reason a good m has come to be used as the general medium of exchange, and follow the fate of those individuals who are born with a non-monetary good i ($\neq$ m) and in need of another non-monetary good j ($\neq$ m and i). Their total frequency is represented by $\bar{e}_{ij}$. In so far as the good m is used as the exclusive medium of exchange, part of these individuals are in (i,m) trading zone as i-supplying m-demanders, i.e., as the sellers of their initial endowment. Their frequency is represented by e_{iij}. The rest are in (m,j) trading zone as the m-supplying j-demanders, i.e., as the buyers of their needful good. Their frequency is represented by e_{imj}, and we have an adding-up equation: $\bar{e}_{ij} = e_{iij} + e_{imj}$. Now, whenever one of the i-endowed j-consumers with good i in hand encounters a buyer of good i in (i,m) zone, he sells his endowment i to the latter and obtains a monetized good m in exchange. He then leaves (i,m) for (m,j) trading zone. Since the probability of such an encounter per time unit is equal to the frequency q_{mi} of the buyers of good i, the total "death" probability of e_{iij} can be calculated as $q_{mi}e_{iij}$. On the other hand, whenever one of i-endowed j-consuers with monetized good m in hand encounters a seller of good j in (m,j) trading zone, he buys his needful good from the latter and retires from the economy. Since the probability of such an encounter per time unit is equal to the frequency q_{jm} of the sellers of good j, the total "death" probability of e_{imj} can be calculated as $q_{jm}e_{imj} = q_{jm}(\bar{e}_{ij} - e_{iij})$. Since we have assumed that as soon as an old i-endowed, j-consumer retires from the economy a new i-endowed, j-consumer enters into it, this also represents the total "birth" probability of e_{iij}. Then the law of large numbers allows us to write down (as an approximation) the "net" change of eiij for i ($\neq$ m) and j ($\neq$ m and i) as:

(12a) $\quad \dot{e}_{iij} \;=\; -q_{mi}e_{iij} + q_{jm}(\bar{e}_{ij} - e_{iij})\,,$

where $\dot{e}_{iij}$ denotes a time-derivative of e_{iij}.

As for those individuals who happen to be born with or in real need for the monetized good m, no indirect trades are necessary to fulfill their needs. We thus have

(12b) $\quad e_{mmj} = \bar{e}_{mi} \quad$ and $\quad e_{iim} \;=\; \bar{e}_{im}\;.$

If we substitute the Clower equations (11a) and (11b) into (12a) and take note of (12b), we finally obtain a set of differential equations which completely describe the interacting evolutions of (N-1)(N-2) transient frequencies e_{iij}, for i (≠ m) and j (≠ m and i).

(13) $\quad \dot{e}_{iij} = -\{\bar{e}_{mi} + \Sigma_{h\neq m}(\bar{e}_{ij} - e_{hhi})\}e_{iij} + (\bar{e}_{jm} + \Sigma_{k\neq m}e_{jjk})(\bar{e}_{ij} - e_{iij})\,.$

In spite of its apparent complexity the above dynamical system turns out to be mathematically well-behaved in the sense that it is formally equivalent to the set of price adjustment equations with indecomposable and weakly gross-substitutable excess demand system in the standard Walrasian theory.[15] It thus has a unique and globally stable steady-state. If we set the left-hand-side of (13) equal to zero, we can obtain the (implicit) expressions for the steady-state transient frequencies $\{e^*_{iij}\}$. For later convenience, we only write down their half-baked expressions by setting the left-hand-side of (12a) equal to zero.

(14) $$e^*_{iij} \;=\; \frac{q^*_{jm}}{q^*_{jm} \;+\; q^*_{mi}}\,\bar{e}_{ij} \qquad \text{for i (≠ m) and j (≠ m and j) :}$$

where q^*_{jm} and q^*_{mi} represent the steady-state supply-demand frequencies that can be defined respectively as $q^*_{jm} = \bar{e}_{jm} + \Sigma_{k\neq m}e^*_{jjk}$ and $q^*_{mi} = \bar{e}_{mi} + \Sigma_{h\neq m}(\bar{e}_{hi} - e^*_{hhi})$. We can give the following interpretation to (14). An i-endowed j-consumer spends the former half of his life in (i,m) trading zone selling his endowment i and the latter half in (m,j) zone buying his needful good j. The expected search period in the first and the second zone are $1/q_{mi}$ and $1/q_{jm}$, and a fraction of the life-time in the first and the second zone are $(1/q_{mi})/(1/q_{mi}+1/q_{jm}) = q_{jm}/(q_{jm}+q_{mi})$ and $(1/q_{jm})/(1/q_{mi}+1/q_{jm}) = q_{mi}/(q_{jm}+q_{mi})$, respectively. When the economy

---

[15] See, for instance, Arrow and Hahn [1971] for the notions of weak gross-substitutability and indecomposability. The proof of the existence, uniqueness and global stability of the steady-state $\{e^*_{iij}\}$ is omitted in the present paper, but it is a simple modification of the one given in Appendix 1 of Iwai [1988] which deals with fiat money equilibrium.

has settled down to a steady-state, the Ergodic theorem in probability theory allows us to identify (with probalibity 1) the time-series fraction $q_{jm}/(q_{jm}+q_{mi})$ with the cross-section fraction of i-endowed j-consumers who are currently selling their endowment in (i,m) trading zone. Equation (14) then follows.

If we substitute (14) as well as (12b) and (12c) into the Clower equations (11a)--(11c), we finally obtain a set of formulae which determine the steady-state supply-demand frequencies.

$$(15a)\qquad q^*_{im} = \bar{e}_{im} + \Sigma_{h\neq m} \frac{q^*_{hm}}{q^*_{hm} + q^*_{mi}} \bar{e}_{ih} \qquad \text{for } i\ (\neq m)\ ;$$

$$(15b)\qquad q^*_{mj} = \bar{e}_{mj} + \Sigma_{k\neq m} \frac{q^*_{mk}}{q^*_{jm} + q^*_{mk}} \bar{e}_{kj} \qquad \text{for } j\ (\neq m)\ ;$$

$$(15c)\qquad q^*_{ij} = 0$$

for i ($\neq$ m) and j ($\neq$ m and i).

Note that while the closed-form forumlae are not generally available for $\{q^*_{ij}\}$, our two special cases allow us to calculate them. In the case of doubly symmetric economy (1a) we have

$$(16a)\qquad q^*_{im} = q^*_{mj} = 1/\{2(N-1)\} \quad \text{and} \quad q^*_{ij} = 0 \qquad \text{for } i\ (\neq m) \text{ and } j\ (\neq m \text{ and } i);$$

and in the case of minimally connected economy (1b) we have

$$(16b)\qquad q^*_{im} = q^*_{mj} = 1/(2N) \qquad \text{for } i \neq m-1 \text{ and } m \text{ and } j \neq m \text{ and } m+1\ ;$$

$$q^*_{m-1,m} = q^*_{m,m+1} = 1/N\ ; \quad \text{and} \quad \text{all the other } q^*ij = 0.$$

We are now in a position to demonstrate the completely universal existence of monetary equilibrium. As was remarked at the end of the last section, Proposition 7 fell short of establishing the bootstrap nature of money in its fullest sense, for it still required two conditions -- the positivity of salability and the positivity of purchasability of the monetized good m, or $q_{im} > 0$ and $q_{mj} > 0$ for all i ($\neq$ m) and j ($\neq$ m). But, let us look at (15a) and (15b), their steady-state expressions. The steady-state salability q^*_{im} now consists not only of the frequency $\bar{e}_{im}$ of the original i-endowed m-consumers but also of the fractions of the frequencies $\bar{e}_{ih}$ of all the other i-endowers who are temporarily demanding good m as the exclusive medium of exchange. By the same token the steady-state purchasability q^*_{mj} now consists not only of the frequency $\bar{e}_{mj}$ of the original m-endowed j-consumers but also of the fractions of the frequencies $\bar{e}_{kj}$ of all the other j-consumers who are temporarily supplying good m as the exclusive medium of exchange. And, these are the very consequence of using the good m as the exclusive medium of exchange! We have thus suceeded in reversing the causal order of the Mengerian theory of money. Even if the "real" salability $\bar{e}_{im}$ and the "real" purchasability $\bar{e}_{mi}$ fail to justify the acceptance of good m as a general medium of exchange, the very trading process which uses that good as the exclusive medium of exchange is bound to lift its salability and purchasability above their "real" values in the long-run. Money can sustain itself soley by its own "bootstraps". We formalize this pure "bootstrap" nature of money by:

Proposition 7: A connected economy always has monetary equilibria. Indeed, it has as many monetary equilibria as the number of goods in it. ◊

(Proof): Let us first note that, since the set of equations (15a) and (15b) defines a continuous mapping of a 2(N-1) dimensional simplex into itself, it has by Brower's fixed point theorem a fixed point or, what comes to the same thing, a steady-state solution $\{q^*_{im}; q^*_{mj}\}$. Our next task is to demonstrate that this steady-state solution indeed constitutes a monetary equilibrium. Because of Proposition 6 what remains to be proved is simply that q^*_{im} and q^*_{mj} are both all strictly positive. The proof proceeds in two steps. First, because of the assumed connectedness of the economy we have a sequence of strictly positive endowment-need frequencies connecting good m and good m+1, another sequence connecting good m+1 and good m+2, ..., passing through N and 1, ..., and finally reaching the Nth sequence connecting good m-1 and good m. Joining these sequences, we can construct a closed loop of strictly positive endowment-need frequencies, $\bar{e}_{nm} > 0$, $\bar{e}_{on} > 0$, $\bar{e}_{po} > 0$, ..., $\bar{e}_{lk} > 0$ and $\bar{e}_{ml} > 0$, such that the set of the connected indices, m, n, o, ..., k, l and m, contains all the N indices at least once. Next, substituting n for i in (15a), we have $q^*_{nm} \geq \bar{e}_{nm}$. Since $\bar{e}_{nm} > 0$ by construction, we have $q^*_{nm} > 0$. Substituting o for i again in (15a), we obtain $q^*_{om} \geq q^*_{nm}/(q^*_{nm}+q^*_{mo})\} \bar{e}_{on}$. Since $\bar{e}_{on} > 0$ by construction and $q^*_{nm} > 0$ as we have just deduced, we have $q^*_{om} > 0$. By repeating the same argument from j = p, ..., k to l, we obtain $q^*_{im} > 0$ for all $i \neq m$. The second condition: $q^*_{mj} > 0$ for all $j \neq m$ can also be proved by applying an analogous argument to (15b). It then follows from Proposition 6 that the economy has a monetary equilibrium with good m used as money.

Moreover, since our choice of good m as money is completely arbitrary, we have indeed established the existence of N different equilibria, each using one of N goods as money. (QED)

We have thus seen that a monetary equilibrium is capable of sustaining itself without any "real" foundations. As long as the economy is connected, and this is no real restriction at all, the very process of monetary circulation creates both the general demand for and the general supply of the monetized good at least in the long-run. Even if there is no "real" demand for it and even if there is no "real" supply of it, this long-run "bootstrap mechanism" endows any good in the economy with all the characteristics that money should have. Money is money simply becaues it is used as money. Or,

> One man is king only because other men stand in the relation of subjects to him. They, on the contrary, imagine that they are subjects because he is king. (Marx [1867])

These observations immediately suggest us the possibility of circulating a totally useless disk of base metal or a totally useless sheet of printed paper (or a mere acknowledgement of the ownership of a large round stone sunken deep in the sea) as money. In fact, our basic model of decentralized exchange can be used to the analysis of the economy with "fiat money" with little modifications, simply by introducing into it a fictitious good which no one needs to consume and no one but the government is able to produce. We, however, leave that analysis to another paper.[16]

8. Welfare comparisons.

Having characterized both barter and monetary equilibrium, we now make a detour and compare them from a normative standpoint. For this purpose, we denote by $V^B{}_{ij}$ and $V^M{}_{ij}$ the expected life-time utility of an i-endowed j-consumer in a barter equilibrium and in a steady-state monetary equilibrium, respectively. We, however, know that not all the economies have a barter equilibrium. And when no bater equilibrium exists, the first notation should be understood as the i-endowed j-consumer's expected life-time utility when everybody in the economy seeks (but not necessarily succeeds in) a barter trade. By (9) we have:

---

[16] See Iwai [1988].

(17a) $V^B_{ij} = u\text{-}b\text{-}c/e_{ji}$ for any i and j ($\neq$i).

In order to express V^M_{ij}, it is convenient to divide the whole population into three groups -- (i) those born with a monetized good m, (ii) those in need of a monetized good m, and (iii) all the others, and calculate their expected life-time utilities separately. By (15a) and (15b) we have:

$$V^M_{im} = u - b - c/[\bar{e}_{im} + \Sigma_{h\neq m} \{q^*_{hm}/(q^*_{hm} + q^*_{mi})\} \bar{e}_{ih}] \qquad \text{for } i\ (\neq m)\ ;$$

(17b) $$V^M_{mj} = u - b - c/[\bar{e}_{mj} + \Sigma_{k\neq m} \{q^*_{mk}/(q^*_{mk} + q^*_{jm})\} \bar{e}_{kj}] \qquad \text{for } j\ (\neq m)\ ;$$

$$V^M_{ij} = u\text{-}2b\text{-}c/[\bar{e}_{im}+\Sigma_{h\neq m} \{q^*_{hm}/(q^*_{hm}+q^*_{mi})\}\bar{e}_{ih}]\text{-}c/[\bar{e}_{mj}+\Sigma_{k\neq m}\{q^*_{mk}/(q^*_{mk}+q^*_{jm})\}\bar{e}_{kj}]$$

for i ($\neq$ m) and j ($\neq$ m and i).

Comparing (17a) with (17b), we see that $V^M_{im} > V^B_{im}$ or that those with a real need for a monetized good m always gain from its circulation as money, because of the consequent increase in its supply in every trading zone. Likewise, we see that $V^M_{mj} > V^B_{mj}$ or that those with an endowment of the monetized good m always gain from its circulation as money. Their gain in expected utility can be regarded as a "seigniorage" accruing to their "natural" ability to supply their endowments as money. As for the relationship between V^M_{ij} and V^B_{ij} for i ($\neq$ m) and j ($\neq$ m and i), the first thing we can say is that if there is no double coincidence of wants or $\bar{e}_{ji} = 0$, barter seeking i-endowed j-consumers have to suffer the misery of autarky or $V^B_{ij} = -\infty$. A transition to or more appropriately an establishment of a monetary equilibrium inevitably improves the welfares of those individuals who have originally nothing to do with the monetized good. Indeed, in the case of the minimally connected economy (1b), the absolute lack of double coincidence of wants binds every individual to the state of autarky, so that the circulation of money is expected to raise the life-time utility of every individual from - ∞ to

(18a) $V^M_{m-1,m} = V^M_{m,m+1} = u\text{-}b\text{-}cN$, and

$V^M_{ij} = u\text{-}2b\text{-}4cN$ for $i \neq m\text{-}1$, m and $j \neq m$, m+1.

It is, however, not possible to say in general whether $V^M_{ij} > V^B_{ij}$ or not. This is particularly so when there exists a barter equilibrium. Monetary equilibrium is not always Pareto superior to barter equilibrium. The question, however, is how general this case is. To see this, let us examine the special economy (1a). Its doubly symmetric endowment-need structure provides the most favorable envoronment to barter equilibrium and by (1a) we have::

(18b) $V^B{}_{ij}$ = u-b-cN(N-1);

As for the steady-state monetary equilibrium, we have by (16a):

(18c) $V^M{}_{im} = V^M{}_{mj}$ = u-b-2c(N-1) and $V^M{}_{ij}$ = u-2b-4c(N-1) for i and j (≠ m).

We can then confirm the general ranking: $V^B{}_{im} < V^M{}_{im}$ and $V^B{}_{mj} < V^M{}_{mj}$. What interests us, however, is whether $V^M{}_{ij}$ - $V^B{}_{ij}$ is positive or negative for i (≠ m) and j (≠ m). In fact, we can easily calculate this difference as c(N-1)(N-4)-b and confirm the observation given above. As long as N ≤ 4 a switch from barter to monetary equilibrium does not improve the welfares of those who have neither endowment of nor need for good m. But, this expression at the same time tells us that as the number of goods becomes sufficiently large (more precisely, as $N > 5/2+\sqrt{9/4+b/c}$), even those individuals end up with gaining from such a switch in the long-run and that monetary equilibrium becomes Pareto superior to barter equilibrium. All in all, even though monetary equilibrium is not always Pareto superior to barter equilibrium, it is likely to be so in a wide variety of economies, especially when the number of goods is sufficiently large.

9. On the difficulty of the natural evolution of monetary equilibria.

Money is ubiquitous, at least potentially, because it can support itself by its own bootstraps. We are thus tempted to jump to the conclusion that: "*therefore*, it is only a matter of time that an economy can attain one of its potential monetary equilibria." But, a potentiality is only a potentiality, and the logic of money and the genesis of money should not be confused. Indeed, we are going to show now that there is a fundamental difficulty in the *laissez-faire* evolution of a monetary economy from the world of barter trades.

For this purpose we have to distinguish the individuals' subjective expectation of supply-demand frequency from its objective value. We denote the former by $q^e{}_{ij}$ and retains q_{ij} to denote the latter. (We assume that such subjective estimate is uniform across individuals. Note, however, that its uniformity does not imply its correctness.) Since the individual search activity has no other choice but to be guided by his subjective expectations about trading opportunities, we now have to rewrite the functional equation (2) and the counting equation (8) explicitly in terms of q^e_{ij}.

(11') $\quad V_{ij} = \underset{k}{\mathrm{Max}}\ [V_{kj} - b - c/q^e_{ki}]$.

(8) $\quad q_{ij} = \sum_h \sum_{\{k:\ Vik = Vjk-b-c/q^e ji\}} e_{hik}$ $\qquad$ for any i and j ($\neq$ i) .

Then, the definition of trade equilibrium needs the third set of conditions which requires the equality between subjective and objective supply-demand frequencies, or

(19) $\quad q^e_{ij} = q_{ij}$ $\qquad$ for any i and j ($\neq$i).

It is for the sake of analyzing the dynamic evolution of the decentralized trading system that we have introduced the conceptual distinction between subjective and objective supply-demand frequencies. And it is for this reason we now have to introduce a certain expectation formation process. Since the intrinsic multiplicity of trade equilibria precludes us from adopting the hypothesis of rational expectations, we have no other choice but to introduce something *ad hoc* into our model of expectation formation process. The simplest and the most *ad hoc* is the static expectations model, which assumes that:

(20) $\quad q^e_{ij}(t+dt) = q_{ij}(t)$,

where t denotes a time and dt (> 0)denotes a small time period. There are of course many other formulations of expectation-formation process, but this is sufficient for our analysis.

Suppose now that an economy has a barter equilibrium and that it indeed finds itself in it. The question is: will the economy be able to evolve naturally from it to one of potential monetary equilibria? Will people's learning process and trading process work together to uproot the economy from the shackle of barter equilibrium and automatically implant it into a monetary equilibrium? In order to answer this question, we first state two trivial observations.

Proposition 8: Each of the possible monetary equilibria is locally stable. ◊

(Proof): A small change in q^e_{ij} preserves (7a)-(7d) in Proposition 3 and hence keeps every q_{ij} unchanged. Under static expectations the economy is immediately sent back to the original monetary equilibrium. (QED)

Proposition 9: Suppose that a barter equilibrium exists and satisfies the inequalities (6) of Proposition 4 strictly. Then, it is locally stable. ◊

(Proof): As long as the inequalities (6) are strict, a small change in q^e_{ij} does not violate them and keep every q_{ij} unchanged. Under static expectations the economy is immediately sent back to the original barter equilibrium. (QED)

Proposition 8 is a good news, for it says that once the economy finds itself in one of possible monetary equilibria, it can not easily be displaced from it. Proposition 9 is, however, a bad news, for it says that as long as disturbances remain small, the economy can never escape from the tyranny of its own "real" conditions which dictate barter equilibrium. Contrary to what Menger said, we can find no "natural" tendency for the evolution from barter to monetary economy.

In the case of doubly symmetric economy (1a) we can illustrate the above two propositions by means of a simple diagram. For this purpose, let us divide the whole population into three groups, and set $q_{ij}(t) = x(t)$ for i (≠ m) and j (≠ m and i); $q_{im}(t) = y(t)$ for i (≠ m); and $q_{mj}(t) = z(t)$ for j (≠ m). Here, x(t) is the representative supply-demand frequency of those individuals who neither supply nor demand good m, y(t) that of those individuals who currently demand good m, and z(t) that of those individuals who currently supply good m. Let us also set their subjective counterparts as $q^e_{ij}(t) = x^e(t)$ for i (≠ m) and j (≠ m and i)m; $q^e_{im}(t) = y^e(t)$ for i (≠ m); and $q^e_{mj}(t) = z^e(t)$ for j (≠ m). Such bundling of objective as well as subjective supply-demand frequencies is tantamount to ignoring local disturbances. Note that these representative variables satisfy the adding-up equations:

(21a) $(N-1)(N-2)x(t) + (N-1)y(t) + (N-1)z(t) = 1$; and

(21b) $(N-1)(N-2)x^e(t) + (N-1)y^e(t) + (N-1)z^e(t) = 1$.

Note also that we have as initial conditions: $x(0) = y(0) = z(0) = x^e(0) = y^e(0) = z^e(0) = 1/\{N(N-1)\}$. They satisfy all the inequalities (6) for the existence of a barter equilibrium.

Suppose now that in period t people become suddenly optimistic about $y^e(t)$ at the expense of $x^e(t)$, while keeping $z^e(t)$ constant. If this disturbance is not large enough to upset the following inequality (which is in fact a restatement of the inequalities (6)):

(22) $b + c/y^e(t) + c/z^e(t) \geq c/x^e(t)$,

individuals still find it less costly to barter with each other, and we have $y(t) = x(t) = 1/\{N(N-1)\}$. Under the hypothesis of static expectations, we then have $y^e(t+dt) = y(t) = 1/\{N(N-1)\}$ and $x^e(t+dt) = x(t) = 1/\{N(N-1)\}$, and the economy returns to the original barter equilibrium.

Suppose, on the other hand, that the disturbance is large enough (and by construction wide enough) to reverse the above inequality. This is tantamount to the fulfillment of inequality (7a) in Proposition 3. (All the other inequalities are automatically satisfied.) Then, every individual, except those who are endowed with or in need of good m, now finds it less costly to use good m as a medium of exchange. This immediately implies y(t) = (N-1){1/N(N-1)} = 1/N, and x(t) = 0. Under the hypothesis of static expectations, we have y^e (t+dt) = 1/N, and x^e (t+dt) = 0. However unfounded on the "real" conditions of the economy, a large enough and wide enough optimism about y^e (t), the general demand for good m, has now triggered the bootstrap mechanism and created the very conditions for its own fulfillment. The economy has now evolved into a monetary equilibrium with good m used as money.

Denote by Y^e the water-shed value of y^e which marks the boundary of inequality (20). Since z^e (t) = 1/{N(N-1)} and x^e (t) = {1/(N-2)} {1/(N-1)-y^e (t)-z^e (t)}, we can calculate this explicitly as

$$(23)\qquad Y^e = \frac{b + \sqrt{b^2 + 4N^2(N-1)c} + bN}{b + 2N^2(N-1)c}$$

Then, we can restate the evolutionary dynamics of y^e (t) in the form of a step function:

$$(24)\qquad \begin{aligned} y^e(t) \le Y^e \quad &\rightarrow \quad y^e(t+dt) = 1/\{N(N-1)\}; \\ y^e(t) > Y^e \quad &\rightarrow \quad y^e(t+dt) = 1/N\,. \end{aligned}$$

Fig. 1 illustrates this in a Cartesian diagram with y^e (t) as abscissa and y^e (t+dt) as ordinate. (Since the choice of monetary good m is totally arbitrary, we could have drawn an identical diagram for every one of N goods in the economy.) A trade equilibrium is an intersection between the step function (24) and the 45 degree line. There are indeed two such equilibria -- y^e = 1/{N(N-1)} corresponds to barter equilibrium and y^e = 1/N monetary equilibrium -- and both are locally stable. If the economy will ever evolve from barter to monetary equilibrium the value of y^e (t) must increase beyond the critical level of Y^e and if the economy will ever evolve back to barter from monetary equilibrium the value of y^e (t) must decrease below the critical level of Y^e.

This is not the end of the story. If the economy stays in one of monetary equilibria long enough, the process of monetary trades will gradually force the economy to approach a steady-state, whose supply-demand frequencies $\{q^*_{im};q^*_{mj}\}$ were already calculated in (16a). It then follows that $y^e(t) = q^e_{im}(t)$ decreases to the long-run value of $q^*_{im} = 1/\{2(N-1)\}$, and $z^e(t) = q^e_{mj}(t)$ increases to $q^*_{mj} = 1/\{2(N-1)\}$. Hence, Y^e should now be replaced by its steady-state value:

$$(25) \qquad Y^{e*} = \frac{-2c(N-1)(N-2)+b+\sqrt{\{2c(N-1)(N-2)-b\}^2+8c(N-1)\{b+2c(N-1)\}}}{4(N-1)\{b+2c(N-1)\}}$$

The dynamics of $y^e(t)$ after the economy has settled down to one of possible monetary equilibria for a long period of time is now written as:

$$(26) \qquad \begin{aligned} y^e(t) \le Y^{e*} \quad &\rightarrow \quad y^e(t+dt) = 1/\{N(N-1), \\ y^e(t) > Y^{e*} \quad &\rightarrow \quad y^e(t+dt) = 1/\{2(N-1)\}. \end{aligned}$$

Fig. 2 illustrates the evolutionary dynamics of $y^e(t)$ when the economy has already established itself in a steady-state monetary equilibrium. Since it is not hard to show that $Y^{e*} < Y^e$, it tells us that the monetary equilibrium has a certain robustness in the sense that in order for the economy to return from it to the original barter equilibrium there needs a disturbance much larger than the one required for the original evolution into it. There is a certain "irreversibility" in the dynamic evolution from barter to monetary equilibrium. Notice, however, that this by no means precludes the possibility that a wide-spread collapse of the confidence on the salabilty of the monetized good may trigger a process equivalent to hyper-inflation and send the monetary equilibrium back to the barter equilibrium.

In any case, what we have seen in Fig. 1 and Fig. 2 is the fundamental difficulty in the "natural" evolution of money and monetary trades in a decentralized exchange economy. Indeed, that difficulty appears to be almost insurmountable in the second special economy of minimally connected structure (1b), though its analysis is omitted in the present paper.

Nonetheless, no matter how difficult the evolution of money might be from the standpoint of pure theory, we cannot at the same time deny the hard fact that we are actually living in a full-fledged monetary economy. What in the world was the large symmetry-breaking disturbance which historically created money and monetary economy "in the beginning"? But such question is certainly better put to the hands of historians, archaeologists and numismatists.[17]

[17] Philip Grierson [1978] has given us the most sophisticated account of the "origins of money". See also A. Quiggin [1949], G. Dalton [1965], P. Einzig [1966], H. Codere [1968] and many others.

10. On money, markets and macroeconomics -- concluding remarks

Money is accepted as money by everybody in the economy simply because it is accepted as money by everybody else in the economy. It is indeed a pure "social entity" whose existence owes nothing to the "technology and preferences" of the economy. But it is precisely for this lack of real foundations that money is able to surmount the requirement of double coincidence of wants for barter trades and make the otherwise impossible trades possible among searching individuals. And the resulting trading system created by such bootstrap mechanism is the one in which everybody offers a good in hand in exchange for a monetized good and receives a good in need in exchange of a monetized good. What we call "markets" where people "sell" and "buy" goods in our daily sense of the words have thus emerged endogenously. Markets as economic institutions are no more than the joint product of the bootstrap mechanism which has produced money as a social entity.

Indeed, once money and markets have come into existence jointly, all the trading zones become defunct, except those involving money! For it is the trading zones between a monetized good and other non-monetary goods that now function as the "markets" for the latter. This leads us to two important observations. First, in a monetary economy every trading individuals are forced to observe the "Clower constraints" which require them to have money in advance whenever they wish to obtain some non-monetary goods. There is no free lunch, and the removal of the "real" constraints of barter trades has now introduced these "monetary" constraints into our decentralized exchange economy, together with all the macroeconomic complications associated with them. Second, the common habit of economists to talk about the "demand and supply of money" as if the conventional demand and supply analysis can be applied to money with little modification is now put into question. Of course, apples have their own market, automobiles have their own market, foreign currencies have their own market, and even promises to pay money in the future have their own market. But, money itself cannot have its "own market" by the very fact that it is used as the medium of exchange in all the other markets in the economy. When there is a disequilibrium between demand and supply of apples or automobiles or foreign currencies or future moneys, that disequilibrium is adjusted directly in apple market or automobile market or foreign exchange market or bond market, respectively. But money has no market of its own, and when there is an disequilibrium between demand and supply of money, that disequilibrium can only be adjusted indirectly through the complex and interacting adjustments in the markets of all the other goods in the economy. In contrast to real disequilibria, monetary disequilibria are by nature macroeconomic phenomena, forcing the entire economy to take part in their adjustment processes.

These observations suggests us new and interesting research agenda for macroeconomic theory as well as macroeconomic policy. Yet, we wish to be the first to point out that the model of decentralized exchange economy developed in the present paper is still too primitive to be of any use for comprehensive macroeconomics. Further works to relax its many restrictive assumptions are very much in need before we embark on such an endeavour.

Appendix -- The Proof of Proposition 3.

The purpose of this appendix is to prove Proposition 3 of the main text. We first prove the sufficiency of (7a) -- (7d). Since the expected utility of using good m as the sole medium of exchange is given as $u-2b-c/q_{jm}-c/q_{mi}$, it is only necessary to show that this value is the unique maximum of V_{ij} for any i($\neq$ m) and j($\neq$ m and i). We need a lemma.

Lemma 1: Neither an individual with an endowment of good m nor an individual with a real need for good m seeks an indirect trade if (7c) holds for any j($\neq$ m) and k($\neq$ m and j).

(Proof): By repeatedly applying (7c) to its own R-H-S, we obtain a series of inequalities: $c/q_{jm} \leq b+c/q_{jk}+c/q_{km} \leq 2b+c/q_{jk}+c/q_{kh}+c/q_{hm} \leq ...$, for any k ($\neq$ m) and h ($\neq$ m) and so on. They of course imply that $V^0_{mj} = u-b-c/q_{jm}$ is the unique maximizer of Vmj. By applying a similar argument to (7d), we can also show that $V^0_{im} = u-b-c/q_{mi}$ is the unique maximizer of V_{im}. (QED)

We are now ready to prove the sufficiency part of Proposition 3. For any i ($\neq$ m) and j($\neq$m and i), (7a) says that $u-2b-c/q_{jm}-c/q_{mi}$ is strictly greater than $V^0_{ij} = u-b-c/q_{ji}$, and (7b) that it is the unique maximum of V^1_{ij}. What remains to be proved is only that it is also strictly greater than V^n_{ij} for any $n \geq 2$. Suppose not. Then, there is a sequence of n ($\geq$ 2) indirect trades which use goods k,..., h as media, such that $u-2b-c/q_{jm}-c/q_{mi} < u-(1+n)b-c/q_{jk}-...-c/q_{lh}-c/q_{hi}$. Then, by applying (7a) to the last term in the R-H-S we have $< u-(2+n)b-c/q_{jk}-...-c/q_{lh}-c/q_{hm}-c/q_{mi}$. By applying one of the inequality in the proof of Lemma 1 to the middle terms, we obtain $u-2b-c/q_{jm}-c/q_{mi} < u-2b-c/q_{jm}-c/q_{mi}$. This is an outright contradiction. (QED)

We now turn to the proof of the necessity part, which requires us to deduce (7a) -- (7d) from the fact that $u-2b-c/q_{jm}-c/q_{mi}$ is the unique maximum of V_{ij}.

Lemma 2: In order for good m to be used as money, neither its holders nor its consumers must not engage in any indirect trade.

(Proof): Suppose that holders of good m engages in an indirect trade. Then, there is a sequence of n (≥1) indirect trades such that $u-(n+1)b-c/q_{jk}-c/q_{kl}-...-c/q_{gh}-c/q_{hm} > u-b-c/q_{jm}$ for some $h \neq m$. By adding $-b-c/q_{mi}$ to both sides, we have $u-(n+2)b-c/q_{jk}-c/q_{kl}-...-c/q_{gh}-c/q_{hm}-c/q_{mi} > u-2b-c/q_{jm}-c/q_{mi}$. But R-H-S of this inequality is V_{ij}, which implies that a sequence of n+1 indirect trades gives a higher expected utility than the supposed maximum V_{ij}. A contradiction. That the consumers of good m must not engage in any indirect trade can be proved analogously. (QED)

We are now able to deduce the necessity part of Proposition 3. First, that $u-2b-c/q_{jm}-c/q_{mi}$ is the unique maximum of V_{ij} implies (7a) and (7b). Next, Lemma 2 implies that $V_{mj} = u-b-c/q_{jm} \geq u-b-\mathrm{Min}_k\,[b+c/q_{jk}+c/q_{km}]$ and $V_{im} = u-b-c/q_{mi} \geq u-b-\mathrm{Min}_k\,[b+c/q_{mk}+c/q_{ki}]$. They in turn imply (7c) and (7d) respectively. (QED)

Bibliography

David Cass and Menahem Yaari [1966], "A re-examination of the pure consumption loans model," Journal of Political Economy, vol. 74, (August).

Robert W. Clower [1967], "A reconsideration of the microfoundations of monetary theory," Western Economic Journal, vol. 6, (December).

Helen Codere [1968], "Money-exchange systems and a theory of money," Man, ns:3, pp. 557-577.

George Dalton [1965], "Primitive money," American Anthropologist, vol.68.

Peter A. Diamond and Eric Maskin [1979], "An equilibrium analysis of search and breach of contract I : steady states," Bell Journal of Economics, vol. 10, (Spring).

------------- [1982], "Aggregate demand management in search equilibrium," Journal of Political Economy, vol. 90, no. 5, (December).

------------- [1984], "Money in search equilibrium," Econometrica, Vol. 52, no. 1, (January).

Paul Einzig [1966], Primitive Money, 2nd ed., Pergamon: New York.

William Feller [1968], An Introduction to Probability Theory and Its Applications: Volume I, 2nd ed., John Wiley: New York.

Philip Grierson [1978], "The origin of money," Research in Economic Anthropology, vol. 1, JAI Press; pp. 1-35.

Frank Hahn [1981], Money and Inflation, Basil Blackwell: London.

Katsuhito Iwai [1988], "Fiat money and aggregate demand management in a search model of decentralized exchanges," CARESS working paper, #88-16, Unversity of Pennsylvania..

William. S. Jevons [1875], Money and the Mechanism of Exchange, London: King, 1875

Robert A. Jones [1976], "The origin and development of media of exchange," Journal of Political Economy, vol.84, no.4, pt. 1.

Nobuhiro Kiyotaki and Randy Wright [1989], "On money as a medium of exchange," Journal of Political Economy, vol. 97, (December).

Karl Marx [1867], Capital, Volume I, 1867, translated by A. Moore and E. Aveling, New York: International Publisher, 1967.

Carl Menger [1871], Principles of Economics, 1871, translated by J. Dingwall and B. F. Hoselitz, New York: New York University Press, 1981.
----------- [1892], "On the origins of money," Economic Journal, vol.2, (March).
Dale Mortensen [1982], "The matching process as a noncooperative bargaining game," in J.J. McCall ed., The Economics of Information and Uncertainty, University of Chicago Press: Chicago.
Jurg Niehans [1978], The Theory of Money, Baltimore: Johns Hopkins University Press.
Seonghwan Oh [1989], "A theory of a generally acceptable medium of exchange and barter", Journal of Monetary Economics, vol. 23.
Joseph M. Ostroy and Ross M. Starr [1975], "Money and the decentralization of exchange," Econometrica, vol. 42, (November).
A. Hingston Quiggin [1949], A Survey of Primitive Money: The Beginnings of Currency, Methuen: London.
Paul A. Samuelson [1958], "An exact consumption-loan model of interest with or without the social contrivance of money," Journal of Political Economy, Vol.66, (December).
Adam Smith [1776], An Inquiry into the Nature and Causes of the Wealth of Nations, 1776, New York: The Modern Library Edition, 1967.
Ross M. Starr [1972], "The structure of exchange in barter and monetary economies," Quarterly Journal of Economics, vol. 86, (May).

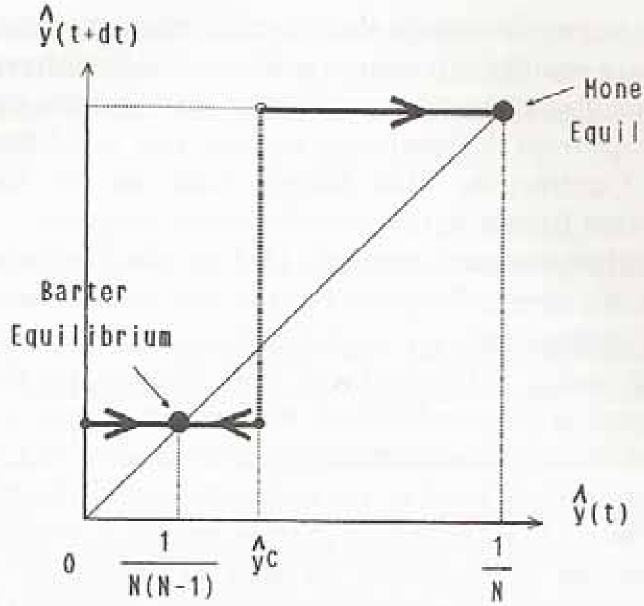

Fig. 1. The evolutionary dynamics of the subjective frequency of the demanders of good *m* in the short run (the case of a doubly symmetric economy).

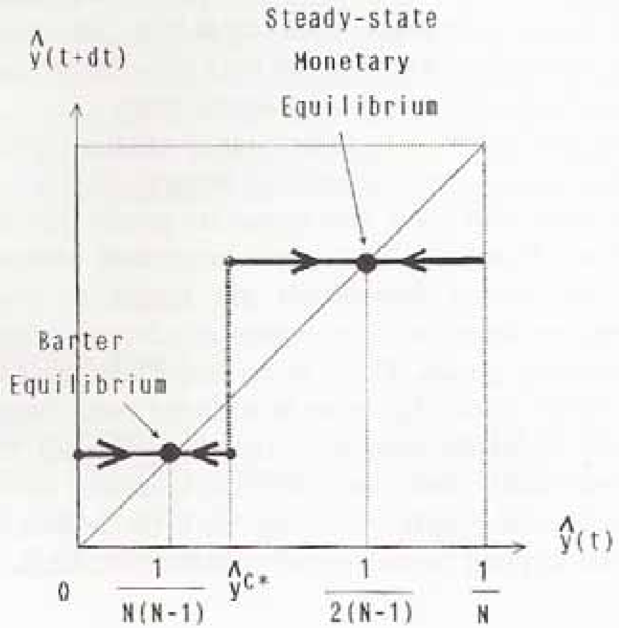

Fig. 2. The evolutionary dynamics of the subjective frequency of the demanders of good *m* in the long run (the case of a doubly symmetric economy).